Efficient Ex Post Implementable Auctions and English Auctions for Bidders with Non-Quasilinear Preferences

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February 9, 2017

Abstract

We study efficient auction design for a single indivisible object when bidders have interdependent values and non-quasilinear preferences. Instead of quasilinearity, we assume only that bidders have positive wealth effects. Our setting nests cases where bidders are ex ante asymmetric, face financial constrains, are risk averse, and/or face ensuing risk. We give necessary and sufficient conditions for when there exists an ex post implementable and (ex post Pareto) efficient mechanism. Necessary and sufficient conditions for the existence of efficient ex post implementable mechanisms differ between the case where the object is a good and when the object is a bad.

In the good setting, there is an efficient ex post implementable mechanism if there is an efficient ex post implementable mechanism in a corresponding quasilinear setting. This result extends established results on efficient ex post equilibria of English auctions with quasilinearity to our non-quasilinear setting. Yet, in the bad setting (i.e. a procurement auction) there is no mechanism that has an ex post efficient equilibrium if the level of interdependence between bidders is sufficiently strong. This result holds even if bidder costs satisfy standard crossing conditions that are sufficient for efficient ex post implementation in the quasilinear setting.

Keywords: Ex post efficient auction, interdependent values, non-quasilinear preferences.
JEL Codes: C70, D44, D47, D61, D82.

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‡The authors thank Dirk Bergemann and Larry Samuelson for helpful comments.

1 Introduction

Efficient auction design is a central question in mechanism design. In the private value single unit quasilinear benchmark case, the English auction has an efficient dominant strategy equilibrium. More recent research gives necessary and sufficient conditions for when the English auction has efficient ex post equilibrium when bidders have interdependent values. Thus, there are well-understood settings where the English auction's efficient equilibrium is robust to asymmetries across bidder beliefs and higher order beliefs.

While the canonical results on English auctions show they are robust to asymmetries in bidder beliefs, these results require strong restrictions on bidder preferences — namely, quasilinearity. Yet in many auction settings, bidders do not have quasilinear preferences, and violations of quasilinearity are frequently cited as being salient to bidders. For example, Maskin (2000) cites financial markets imperfections and liquidity constraints as constraining bid behavior. Salant (1997) draws on his personal consulting experience to argue that financial constraints are a salient feature of how bidders determine their bids. In addition to access to credit, risk aversion and wealth effects are important features of auctions for larger items like houses.[1]

In this paper, we study the efficient auction design problem for a single indivisible unit when bidders have interdependent values. We remove the quasilinearity restriction on bidder preferences, and assume only that bidders have weakly positive wealth effects. We do not place functional form restrictions on bidder preferences, and our setting is sufficiently general to allow for asymmetric bidders who are risk averse, have financing constraints, have budgets, or face ensuing risk. We give necessary and sufficient conditions under which we can construct an auction that has desirable incentive and efficiency properties. Interestingly, the necessary and sufficient conditions for efficient auction design differ between a setting where the object is a good and where the object is a bad. When the object is a good, we show that sufficient conditions for the existence of an efficient ex post equilibrium in the English auction with quasilinearity can be extended to the non-quasilinear preference domain. In particular, we show that the English auction has an efficient ex post equilibrium in the non-quasilinear setting if an English auction has an efficient ex post equilibrium in a corresponding quasilinear setting. Bidder values in the corresponding quasilinear setting are set equal to willingness to pays in the non-quasilinear setting. Yet, when the object being sold is a bad (i.e. a procurement auction), we show the English auction does not have an efficient ex post equilibrium when the level of interdependence among bidders is sufficiently strong. This is distinct from results in the quasilinear setting that give conditions under which efficient design is possible

[1] Homes are often sold via auction. For example, in Melbourne, Australia an estimated 25-50% of homes are sold via auction (see Mayer (1998)).

even if bidders have strong interdependence (see, for example, Krishna (2003)).

Removing quasilinearity complicates the efficient auction design problem. With quasilinearity, an auction outcome is efficient if and only if the bidder with the highest value wins (or lowest cost in a procurement setting). Hence, the space of efficient allocations is independent of bidder transfers. Yet, in our non-quasilinear setting a losing bidder's willingness to pay for the object and the winning bidders' willingness to sell the object both depend on the amount they paid (or were paid) in the auction. Thus, the space of (ex post Pareto) efficient outcomes depends on both the allocation of the object and bidder transfers.[2]

While the space of efficient outcomes is qualitatively different when we remove quasilinearity, we show that canonical results on efficient ex post equilibria of auctions with quasilinearity can be used to study efficient design in our non-quasilinear setting. When the object being sold is a good, a mechanism has an efficient ex post equilibrium in our non-quasilinear setting if the mechanism has an efficient ex post equilibrium in a corresponding quasilinear setting (Theorem 1).[3] This means that a mechanism is efficient if it assigns the good to the bidder with the highest willingness to pay for the good. To understand this result, note that individual rationality implies that the winning bidder pays an amount that does not exceed her willingness to pay for the good. Thus, winning the good makes the winning bidder feel wealthier — it is as though she pays her willingness to pay to win the good, and then is given a partial refund. Positive wealth effects imply that refund increases the winning bidder's willingness to sell the unit (conditional on winning) relative to her willingness to pay for the good. That is, the winning bidder's willingness to sell the good conditional on her payment exceeds her willingness to pay for the good before making any payments. Moreover, the mechanism assigns the good to the bidder with the highest willingness to pay. Hence, the winning bidder's willingness to sell the unit exceeds any of her rivals' willingness to pay for the good and there are no ex post Pareto improving trades among bidders.

Theorem 1 also implies that an English auction has an efficient ex post equilibrium in the non-quasilinear setting if the English auction has an efficient ex post equilibrium in a corresponding quasilinear setting where a bidder's value equals her willingness to pay in the non-quasilinear setting. Thus, sufficient conditions for the existence of an ex post efficient equilibrium in English auctions with quasilinearity extend to our non-quasilinear setting.

[2] In keeping with the prior literature, we (1) use the Myerson and Holmstrom (1983)'s notion of ex post Pareto efficiency, as our efficiency criterion and (2) we look only at the space of deterministic mechanisms and outcomes. Both points are discussed in greater detail in section 2.

[3] In a similar manner, Che and Gale (2006) study single unit auctions for bidders private values and non-quasilinear preferences. They construct a corresponding quasilinear setting to characterize bid behavior in standard auctions. More recently, Nöldeke and Samuelson (2015) study a two sided matching problem without quasilinearity. Similar to this paper, they extend well-known results from the quasilinear domain to a more general preference domain.

That is, the English auction has an efficient ex post equilibrium in the non-quasilinear setting if bidder willingness to pays satisfy the crossing conditions established by Maskin (1992), Krishna (2003), or Birulin and Izmalkov (2011).

We get distinct results when we study a procurement auctions for bidders with positive wealth effects. In the procurement setting, Theorem 2 shows that efficiency in a corresponding quasilinear setting is a necessary condition (but not sufficient) for efficiency without quasilinearity. That is, the auction is efficient only if the bidder with the lowest reservation cost is assigned the task. The intuition behind Theorem 2 mirrors the intuition behind Theorem 1. Individual rationality implies that the winning bidder is paid an amount that weakly exceeds her reservation cost. This makes the winning bidder feel richer and then positive wealth effects implies that she is willing to offer one of her rivals a higher amount on money (relative to her reservation cost) to complete the task on her behalf. Thus, if the winning bidder is not the lowest cost bidder, there is Pareto improving trade where the winning bidder pays the lowest cost bidder to complete the procurement task on her behalf. Hence, efficiency requires that the bidder with the lowest reservation cost wins the good.

We use Theorem 2 to show that there is no mechanism that has desirable incentive and efficiency properties when the degree of interdependence among bidder's is sufficiently strong. We illustrate this result in a symmetric setting like that presented by Milgrom and Weber (1982), but without quasilinearity. Signals are additive and α measures the degree of interdependence among bidders. In this setting the English auction has an ex post equilibrium where the lowest cost bidder wins the task. Yet, when α is sufficiently close to one, then there is no mechanism that satisfies (1) individual rationality, (2) ex post incentive compatibility, (3) no subsidies, and (4) efficiency (Theorem 3). This is because the winning bidder must be paid an amount that exceeds her reservation cost to complete the procurement task. This makes the winning bidder richer. Thus, positive wealth effects imply that the winning bidder is willing to offer another bidder an amount that exceeds her reservation cost to complete the procurement task. In addition, when α is sufficiently to one, the losing bidders have reservation costs that are close to the winning bidder's reservation cost. Hence, when α is sufficiently large, there is a Pareto improving trade where the winning bidder pays her rival to complete the procurement task on her behalf.

Related Literature

Prior work on efficient auction design without quasilinearity has primarily focused on private value settings. Much of this work studies revenue maximizing auctions, but in this paper, we study the efficient design problem.[4] More toward our focus, Maskin (2000), Saitoh and Serizawa (2008), Morimoto and Serizawa (2015), and Baisa (2016) show that efficient auction design is possible only if bidders have single unit demand or single-dimensional types. In this paper, we study the single unit model and we provide necessary and sufficient conditions under which efficient auction design is possible. Yet, in contrasts to the the aforementioned papers where bidders have private values, we show there relevant economic settings — specifically procurement settings — where there is no auction with an ex post efficient equilibrium, even when bidders have single dimensional types that satisfy established single crossing conditions.[5]

There are fewer papers that study auctions with the interdependent value and non-quasilinear preferences. The exceptions are Fang and Parreiras (2002, 2003) and Hu, Matthews, and Zou (2015). The Fang and Parreiras papers study revenue implications of information disclosure in English auctions when bidders have budgets and interdependent values. The focus of Hu, Matthews, and Zou (2015) is closer to our paper. They study the efficiency properties of the English auctions when bidders are risk averse and face ensuing risk. In Section 4 we show that ensuing risk can also be nested in our normal good setting, and we provide an alternative interpretation of their efficiency results can be developed by considering bidder wealth effects.

The remainder of the paper proceeds as follows. Section 2 introduces our formal model. Section 3 shows the relationship between ex post implementation without quasilinearity and ex post implementation in a corresponding quasilinear setting. Section 4 gives results on the normal good setting, and section 5 then studies the procurement setting without quasilinearity.

2 Model

2.1 Preferences

A seller has a single indivisible object and there are $1, \ldots, n$ bidders. Bidder i receives a private signal $s_i \in S_i \subset \mathbb{R}^k$.[6] For simplicity we let $S^n := \times_{i=1}^n S_i$. We assume bidders are

[4] For example, Maskin and Riley (1984), Pai and Vohra (2014), and Baisa (2017) all show there are important qualitative differences in the revenue maximization problem when we remove quasilinearity.

[5] Relatedly, Hu, Matthews, and Zou (2015) present an interdependent value setting where bidders face ensuing risk and the English auction does not have efficient ex post equilibria. This can occur when bidders have decreasing absolute risk aversion.

[6] We will often study the case where a bidder's signal space is one-dimensional $S_i \subset \mathbb{R}$. Indeed, there are many impossibility results already in the quasilinear setting on efficient implementation with multi-dimensional types (see Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Jehiel et al. (2006)). This is discussed in greater detail at the end of Section 3.

utility maximizing and bidder i's preferences are described by utility functions u_i^x where

$$u_i^x : \mathbb{R} \times S^n,$$

and $x \in \{0,1\}$ indicates whether the bidder has the good or not. Bidder i gets utility $u_i^x(t,s)$ when she wins $x \in \{0,1\}$ objects, receives transfer $t \in \mathbb{R}$, and bidders $1,\ldots,n$ have signals $s := (s_1,\ldots,s_n) \in S^n$.[7] We assume that a bidder's utility is strictly increasing, continuous, and differentiable in money. That is, $u_i^x(\cdot,s)$ is strictly increasing, continuous, and differentiable $\forall x \in \{0,1\},\ s \in S^n$.

We say that the object being sold is a good if

$$u_i^1(t,s) \geq u_i^0(t,s)\ \forall i \in \{1,\ldots,n\},\ t \in \mathbb{R},\ s \in S^n.$$

When studying the good setting, we assume that bidders have finite demand for the unit. Thus, there exists a $\rho > 0$ such that

$$u_i^0(0,s) > u_i^1(-\rho,s)\ \forall i \in \{1,\ldots,n\},\ s \in S^n.$$

Similarly, we say the object being sold is a bad (i.e. a procurement setting) if

$$u_i^0(t,s) \geq u_i^1(t,s)\ \forall i = 1,\ldots,n;\ t \in \mathbb{R};\ s \in S^n.$$

When studying the procurement setting, we assume that bidder is willing to accept the bad (i.e. complete the task) for a finite amount of money. Thus, there exists a $\rho > 0$ such that

$$u_i^1(\rho,s) > u_i^0(0,s)\ \forall i \in \{1,\ldots,n\},\ s \in S^n.$$

We assume that bidders have positive wealth effects. We define positive wealth effects in both the good and bad settings.

Definition 1. (Positive wealth effects for goods)

If the object being sold is a good, then bidders have weakly positive wealth effects if

$$u_i^1(t-p,s) \geq u_i^0(t,s) \implies u_i^1(t'-p,s) \geq u_i^0(t',s)\ \forall t' > t,\ i \in \{1,\ldots,n\},\ s \in S^n,$$

[7]Note, that it is without loss of generality to assume that a bidder begins with initial wealth normalized to zero. If a bidder has preferences $\hat{u}_i$ and initial wealth w_i, then we can define u_i has being $u_i^x(t,s) = \hat{u}_i^x(t+w_i,s)\ \forall x \in \{0,1\},\ t \in \mathbb{R},\ s \in S^n$. Alternatively, a bidder's initial wealth could be a dimension of her private information (see Example 4).

and strictly positive wealth effects if

$$u_i^1(t-p,s) \geq u_i^0(t,s) \implies u_i^1(t'-p,s) > u_i^0(t',s)\ \forall t' > t,\ i \in \{1,\ldots,n\},\ s \in S^n.$$

The definition implies that a bidder's demand for the good is weakly increasing as she becomes wealthier. We analogously define positive wealth effects in the procurement setting.

Definition 2. (Positive wealth effects in the procurement setting)
If the object being sold is a bad, then bidders have weakly positive wealth effects if

$$u_i^0(t-p,s) \geq u_i^1(t,s) \implies u_i^0(t'-p,s) \geq u_i^1(t',s)\ \forall t' > t,\ i \in \{1,\ldots,n\},\ s \in S^n,$$

and strictly positive wealth effects if

$$u_i^0(t-p,s) \geq u_i^1(t,s) \implies u_i^0(t'-p,s) > u_i^1(t',s)\ \forall t' > t,\ i \in \{1,\ldots,n\},\ s \in S^n.$$

The above definition implies that if a bidder is assigned the procurement task, then she would be willing to pay more money to hire a rival to complete the task when she is richer.

It is useful to denote a bidder's willingness to pay and willingness to sell a unit of the good.[8] We let $d_i(t,s)$ be bidder i's willingness to pay for a unit if bidders $1,\ldots,n$ have signals $s = (s_1,\ldots,s_n) \in S^n$ and bidder i received transfers $t \in \mathbb{R}$ prior to her purchasing decision. More formally $d_i(t,s)$ is implicitly defined as

$$d_i(t,s) = d_i\ s.t.\ u_i^1(t-d_i,s) = u_i^0(t,s).$$

We will often be interested in a bidder's willingness to pay prior to receiving any payment from the auctioneer $d_i(0,s)$. For simplicity we write this as $\hat{d}_i(s)$.

Similarly, bidder i's willingness to sell the unit is $h_i(t,s)$ where

$$h_i(t,s) = h_i\ s.t.\ u_i^1(t,s) = u_i^0(t+h_i,s).$$

Our assumptions imply that:

1. $d_i(\cdot,s)$ and $h_i(\cdot,s)$ are continuous and differentiable.

2. In the good setting, $d_i(t,s), h_i(t,s) \geq 0\ \forall s \in S^n,\ t \in \mathbb{R}$.

3. In the procurement setting, $0 \geq d_i(t,s), h_i(t,s)\ \forall s \in S^n,\ t \in \mathbb{R}$.

[8]With quasilinearity, both quantities equal a bidders valuation.

4. If bidders have weakly positive wealth effects $|d_i(\cdot, s)|$ and $|h_i(\cdot, s)|$ are weakly increasing $\forall s \in S^n$.

5. If bidders have strictly positive wealth effects $|d_i(\cdot, s)|$ and $|h_i(\cdot, s)|$ are strictly increasing $\forall s \in S^n$.

2.2 Mechanisms

By the revelation principle, it is without loss of generality to consider direct revelation mechanisms. A direct revelation mechanism maps bidder signals to an outcome. An outcome specifies a feasible allocation of the unit and transfers. An allocation of goods is feasible if $q \in \{0,1\}^n$ and $\sum_{i=1}^n q_i \leq 1$. We let Q be the set of all feasible allocations and $\hat{Q}$ be the set of all feasible allocations where the unit is assigned to a bidder $\sum_{i=1}^n q_i = 1$. A (deterministic) allocation rule q maps the profile of reported signals to a feasible allocation $q : S^n \to Q$.[9] The transfer rule t maps the profile of reported signals to transfers $t : S^n \to \mathbb{R}^n$. A direct revelation mechanism Γ consists of an allocation rule and a transfer rule.

A mechanisms satisfies ex post individual rationality if a bidder is made no worse off by participating.

Definition 3. (Ex post individual rationality)

A mechanism Γ is ex post individually rational (IR) if $\forall s \in S^n$ and $i \in \{1, \ldots, n\}$,

$$u_i^{q_i(s)}(t_i(s), s) \geq u_i^0(0, s).$$

A mechanism is ex post incentive compatible if truthful reporting is always a Nash equilibrium of game where the signal realization $(s_1, \ldots, s_n)$ is common knowledge.

Definition 4. (Ex post incentive compatibility)

A mechanism Γ is ex post incentive compatible (EPIC) if for $\forall s \in S^n$, $i \in \{1, \ldots, n\}$, and $\forall s' \in S^n$ such that $s'_{-i} = s_{-i}$,

$$u_i^{q_i(s)}(t_i(s), s) \geq u_i^{q_i(s')}(t_i(s'), s).$$

[9]We limit attention to auctions that have deterministic outcomes. This is assumption is partly motivated by practical concerns — randomization is rarely used in practice. Moreover, from a theoretical perspective, prior research shows that there is no mechanism that has desirable incentive and efficiency properties when we allow for randomization. When we allow for stochastic mechanisms, efficient auction design for a single unit when bidders are non-quasilinear is equivalent to efficient auction design in a multi-unit setting. Bidder's have downward sloping demand for additional probability units (see Baisa (2017)). Moreover, Baisa (2016) shows that efficient auction design in a multi-unit setting is generally impossible even with private values. Hence, the restriction to deterministic mechanisms is a natural second-best analysis. Furthermore, related literature on efficient auctions without quasilinearity similarly restricts attention to deterministic allocation rules (see Saitoh and Serizawa (2008), Morimoto and Serizawa (2015), and Hu, Matthews, and Zou (2015)).

With quasilinearity an outcome is efficient if the bidder with the highest value gets that good. This is equivalent to saying that there are no ex post Pareto improving trades among bidders. However, without quasilinearity, these two efficiency notions are distinct. We study mechanisms that satisfy the weaker (more permissive) efficiency notion — ex post Pareto efficiency.[10]

Definition 5. (Ex post Pareto Efficient)

Fix $s = (s_1, \ldots, s_n)$. An outcome $(q, t) \in Q \times \mathbb{R}^n$ is ex post Pareto efficient if $q \in \hat{Q}$ and for any $(\tilde{q}, \tilde{t}) \in \hat{Q} \times \mathbb{R}^n$ such that

$$u_i^{\tilde{q}_i}(\tilde{t}_i, s) > u_i^{q_i}(t_i, s)$$

then either

$$u_j^{\tilde{q}_j}(\tilde{t}_j, s) < u_j^{q_j}(t_j, s)$$

for some $j \neq i$ or

$$\sum_{i=1}^{n} t_i > \sum_{i=1}^{n} \tilde{t}_i.$$

The definition says that an outcome is ex post Pareto efficient if any other outcome that makes one bidder strictly bidder off necessarily decreases revenue or makes another bidder strictly worse off. In the good setting it is without loss of generality to restrict attention to feasible allocations where the good is always assigned. Any outcome where the good is unassigned is Pareto dominated by an outcome where the good is assigned. Yet, in the procurement setting, we assume that efficiency requires that some bidder be assigned the object (task). That is, the auctioneer requires that the task is completed. If we do not impose this restriction in the procurement setting, we can trivialize the problem by never assigning the task. We say that a mechanism satisfies ex post Pareto efficiency (hereafter, efficiency) if the outcome $(q(s), t(s))$ is an ex post Pareto efficient outcome for all type realizations $s \in S^n$.

Finally, much of our analysis studies mechanisms that satisfy a no subsidy condition. A mechanism satisfies the no subsidy condition if a bidder does not receive a positive amount money from the auctioneer if she does not win the object.

Definition 6. (No subsidies)

A mechanism satisfies the no subsidy condition if $q_i(s) = 0 \implies t_i(s) \leq 0 \; \forall s \in S^n$.

[10] This is the same efficiency notion used when Dobzinski, Lavi, and Nisan (2012) and Morimoto and Serizawa (2015) study related efficient auction design problems without quasilinearity. It is trivial to show that there are no ex post Pareto improving trades when an outcome that maximizes surplus, yet many have shown that there are outcomes that do not maximize total surplus, yet there are no Pareto improving trades among bidders.

When combined with individual rationality no subsidies gives that a losing bidder makes no transfers $q_i(s) = 0 \implies t_i(s) = 0$.

3 Necessary and Sufficient Conditions for Ex Post Incentive Compatibility

In this section, we establish necessary and sufficient conditions for a mechanism to satisfies (1) IR and (2) EPIC. We show that mechanism Γ satisfies IR and EPIC without quasilinearity if and only if mechanism Γ satisfies EPIC and IR in a corresponding quasilinear setting (Proposition 1). The result holds in the good setting and the procurement setting. In section 4, we use Proposition 1 to derive sufficient conditions for constructing individually rational mechanisms efficient and ex post implementable. We also use Proposition 1 to show that an English auction has an efficient ex post equilibrium if it has an efficient ex post equilibrium in the corresponding quasilinear setting. Then, in Section 5, we use Proposition 1 to give necessary and sufficient conditions under which there exists a mechanism has desirable incentive and efficiency properties in the procurement setting.

The taxation principle (see Rochét (1985)) states that if a mechanism satisfies EPIC, then a bidder's payment depends on (1) whether or not she wins, and (2) her rivals' reported types. Thus, a bidder's payment is constant conditional on the number of units that she wins and her rivals' types reports. Let τ_i^0 is the amount that bidder i pays when she does not win any units. More precisely

$$\tau_i^0(s_{-i}) := \begin{cases} -t_i(s_i, s_{-i}) & \text{if } \exists s_i \in S_i \text{ s.t. } q_i(s_i, s_{-i}) = 0 \\ 0 & \text{if } \nexists s_i \in S_i \text{ s.t. } q_i(s_i, s_{-i}) = 0. \end{cases}.$$

Note that if Γ satisfies the no subsidy condition then $\tau_i^0(s_{-i}) = 0 \ \forall s_{-i} \in \times_{j \neq i} S_j$. Similarly, let τ_i^1 be the marginal price bidder i pays to win a unit,

$$\tau_i^1(s_{-i}) = \begin{cases} \infty & \text{if } \nexists s_i \in S_i \text{ s.t. } q_i(s_i, s_{-i}) = 1 \\ -t_i(s_i, s_{-i}) + \tau_i^0(s_{-i}) & \text{if } \exists s_i \in S_i \text{ s.t. } q_i(s_i, s_{-i}) = 1 \end{cases}.$$

Lemma 1 shows that a mechanism satisfies EPIC and IR if and only if (1) a bidder pays a non-positive price when she does not win the unit ($\tau_i^0(s_{-i}) \leq 0$) and (2) a bidder wins the unit if and only if her demand for her the unit (conditional on paying $\tau_i^0(s_{-i})$ to participate) exceeds the price she needs to pay to win her first unit $\tau_i^1(s_{-i})$.

Lemma 1. *A mechanism* Γ *satisfies EPIC and IR if and only if* $\tau_i^0(s_{-i}) \leq 0$ *and*

$$q_i(s) = 0 \implies \tau_i^1(s_{-i}) \geq d_i(-\tau_i^0(s_{-i}), s),$$

and

$$q_i(s) = 1 \implies d_i(-\tau_i^0(s_{-i}), s) \geq \tau_i^1(s_{-i}).$$

Note that when the mechanism satisfies no subsidies (and hence, $\tau_i^0(s_{-i}) = 0$), then the condition simplifies to stating that $q_i(s) = 1 \iff \hat{d}_i(s) \geq \tau_i^1(s_{-i})$. In other words, bidder i wins the unit if her willingness to pay for the exceeds the price of the unit.

We use Lemma 1 to show a mechanism Γ that satisfies EPIC and IR in the non-quasilinear setting if and only if the mechanism has a corresponding quasilinear setting where the mechanism Γ also satisfies EPIC and IR. We construct this corresponding quasilinear setting — the $QL(\Gamma)$ setting — to be such that bidder i's value equals her willingness to pay for the unit, conditional on having paid $\tau_i^0(s_{-i})$ to participate. Thus, bidder i has value function $v_i : S^n \to \mathbb{R}$, where v_i is defined as,

$$v_i(s) = d_i(-\tau_i^0(s_{-i}), s).$$

Definition 7. A mechanism Γ that satisfies EPIC has a corresponding quasilinear setting $QL(\Gamma)$ where bidders have quasilinear preferences and value function defined as

$$v_i(s) = d_i(-\tau_i^0(s_{-i}), s) \; \forall s = (s_i, s_{-i}) \in S^n.$$

Note that the corresponding quasilinear setting varies with Γ because a bidder's willingness to pay for the object varies with how much she is paid to participate $\tau_i^0(s_{-i})$. If we impose individual rationality and no subsidies, then $\tau_i^0(s_{-i}) = 0 \; \forall s_{-i} \in \times_{j \neq i} S_i$. Thus, the when we restrict attention to mechanisms that satisfy IR and no subsidies, then a mechanism's corresponding quasilinear setting is independent of Γ.

Remark 1. Bidder i has valuation $v_i(s) = \hat{d}_i(s) \; \forall s \in S^n$ in the corresponding quasilinear setting of any mechanism that satisfies IR and no subsidies.

Thus, the corresponding quasilinear setting is independent of the mechanism when we consider IR and no subsidy mechanisms.

Proposition 1. *The direct revelation mechanism* Γ *satisfies EPIC and IR in the non-quasilinear setting if and only if* Γ *satisfies EPIC and IR in setting* $QL(\Gamma)$.

Hence the Proposition states the we can determine if a proposed mechanism satisfies EPIC and IR by looking at the mechanisms corresponding quasilinear setting. If the mechanism satisfies EPIC and IR in the corresponding quasilinear setting, then it satisfies EPIC and IR in the quasilinear setting, and visa versa.

The formal proof is in the appendix, but the intuition is straightforward. We construct the corresponding quasilinear setting to be such that the bidder with non-quasilinear preferences has the same preference ranking over any two outcomes of mechanism Γ as the corresponding quasilinear bidder. That is, the non-quasilinear demands a unit for price $\tau_i^1(s_{-i})$ if and only if the corresponding quasilinear bidder demands a unit for price $\tau_i^1(s_{-i})$.

Note that we will often study the case where a bidder's signal space is one-dimensional $S_i \subset \mathbb{R}$. Indeed, there are many impossibility results in the quasilinear setting on implementation problems with multi-dimensional types (see Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Jehiel et al. (2006)). In addition, all of our examples will also impose that a bidder's willingness to pay for the object is monotone in her signal. Yet neither of these conditions are assumed in Proposition 1 or the Theorems which follow from it in the sections below. This is because our general results do not require that we put structure on the signal space. Instead, our main results will show that we can study a non-quasilinear setting by considering a corresponding quasilinear setting. Our contribution is to show how existing tools from the quasilinear setting can be use to study the non-quasilinear setting. Hence, the dimensionality restrictions on the signal space that are needed for efficient implementation are the same conditions that would be needed on the corresponding quasilinear setting. For example, if we study a setting where signal space is multi-dimensional and ex post implementation of nontrivial social choice functions is impossible in a mechanism's corresponding quasilinear setting, then Proposition 1 similarly shows that the social choice function is not ex post implementable in our non-quasilinear setting.

4 The Normal Good Setting

In this section, we study ex post efficient mechanisms in the good setting where bidders have positive wealth effects (i.e., the good is a normal good). We show that a mechanism is efficient and ex post implementable if the mechanism is efficient and ex post implementable in the corresponding quasilinear setting (Theorem 1). We then use Theorem 1 and recent results in the literature on efficient ex post equilibria of English auctions in the quasilinear setting to obtain analogous sufficient conditions for the existence of efficient ex post equilibria of English auctions in the non-quasilinear setting (Corollary 1).

With quasilinearity, a mechanism is efficient if it assigns the good to the bidder with the highest value. Therefore, the space of efficient outcomes is characterized by the assignment of the good, not by transfers. In particular, an outcome is efficient if and only if the bidder with the highest valuation (conditional on all signal realizations) wins the good. Without quasilinearity, the space of efficient outcomes is distinct. An outcome is efficient if there are no ex post Pareto improving trades. Hence, the willingness to sell good for the winner must exceeds the highest willingness to pay of her rivals. Yet, because of positive wealth effects, the winning bidder's willingness to sell the good declines as she pays more money to the auctioneer. Thus, it can be the case that bidder i's willingness to sell the good is greater than her rivals' willingness to pays for the good if bidder i if and only if bidder i makes a sufficiently low payment. Therefore, we can construct efficient outcomes where bidder i is assigned the good and makes a low payment, and efficient outcomes in which one of bidder i's rivals is assigned the good instead. That is, without quasilinearity, the space of efficient outcomes can included outcomes with distinct assignments. Moreover, a particular assignment can be associated with an efficient outcome for some levels of transfers, but not for other levels of transfers. Both points are important distinctions for the quasilinear domain.

Remark 2 below states that a mechanism Γ is efficient if and only if there are no ex post Pareto improving trades between the winning bidder and a losing bidder.[11]

Remark 2. If Γ satisfies EPIC and IR, then Γ is efficient if and only if

$$h_i(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) \geq d_j(-\tau_j^0(s_{-j}), s), \ \forall s \text{ s.t. } q_i(s) = 1 \text{ and } q_j(s) = 0.$$

While the space of efficient outcomes in quasilinear setting is distinct from the space of efficient outcomes in the normal good setting, Theorem 1 shows that efficient auction design in the quasilinear setting is related to efficient design in the normal good setting. More precisely, if Γ satisfies (1) EPIC, (2) IR, and (3) efficiency in setting $QL(\Gamma)$, then Γ satisfies (1)–(3) in the normal good setting.

Theorem 1. *A direct revelation mechanism* Γ *satisfies (1) IR, (2) EPIC and (3) efficiency in the non-quasilinear setting if* Γ *satisfies EPIC, IR, and efficiency in the corresponding quasilinear setting.*

Before discussing the intuition behind the proof, note that Proposition 1 shows that if Γ satisfies (1) EPIC and (2) IR in setting $QL(\Gamma)$, then Γ satisfies (1) and (2) in the normal good setting. Thus, the proof of Theorem 1 shows that if Γ satisfies (1) EPIC, (2) IR, and (3) efficiency in setting $QL(\Gamma)$, then Γ satisfies efficiency in the normal good setting.

The intuition for the result is straightforward. Consider a mechanism that satisfies the no subsidy condition.[12] Therefore, a losing bidder makes zero payment $\tau_i^0(s_{-i}) = 0 \ \forall s_{-i}$.

[11]While this section studies the normal good setting, this Remark also holds in the procurement setting that we study in Section 5.

[12]The formal proof for a general class of mechanisms is in the appendix.

If mechanism Γ is efficient in the corresponding quasilinear setting, then the mechanism assigns the good to the bidder with the highest willingness to pay. Thus the allocation rule is such that $q_i(s) = 1 \implies \hat{d}_i(s) \geq \hat{d}_j(s)\ \forall j \neq i$. If a bidder wins the good and pays her willingness to pay for the good, then her willingness to sell the good after the auction equals her willingness to pay, because $\hat{d}_i(s) = h_i(-\hat{d}_i(s), s)$. Thus after buying at her willingness to pay $\hat{d}_i(s)$, the bidder is indifferent between selling and keeping the good at this price. Yet, the winning bidder pays a price p where p is less than her willingness to pay to win the good. Since bidders have positive wealth effects,

$$p \leq \hat{d}_i(s) \implies \hat{d}_i(s) = h_i(-\hat{d}_i(s), s) \leq h_i(-p, s),$$

where the first equality holds by the construction of d_i and h_i and the final inequality holds because of positive wealth effects. Therefore, the winning bidder's willingness to sell the object after winning and paying p exceeds her willingness to pay for the good. Furthermore, the winning bidder has the highest willingness to pay of all bidders. Thus, the winning bidder's willingness to sell the good exceeds each of her rival's willingness to pay for the good. Hence, there are no ex post Pareto improving trades of the good.

Theorem 1 is a sufficient — but not necessary — condition to ensure a mechanism satisfies Properties (1)–(3). This is illustrated via Example 5 in the appendix. In the example, there are two ex-ante symmetric bidders, and bidder 1 always wins the good for free. While bidder 1 does not necessarily have the highest willingness to pay, bidder 1's willingness to sell the good is sufficiently large (relative to her rival's willingness to pay for the good) because she wins the good for a low price (free). Thus, in our example, there are no ex post Pareto improving trades, even though the bidder with the highest willingness to pay may not win the good.

4.1 Application: English Auctions

Theorem 1 extends results on the efficiency of English auctions with quasilinearity to the non-quasilinear setting. Note that the English auction satisfies the no subsidy condition by construction. Thus, we have that $\tau_i^0(s_{-i}) = 0\ \forall s_{-i}$. Therefore, Theorem 1 states that the English auction has an efficient ex post equilibrium if there is an efficient ex post equilibrium of the corresponding quasilinear setting where bidder i has valuation $v_i(s) = \hat{d}_i(s)\ \forall s \in S^n$.

Corollary 1. *The English auction in the normal good setting has an efficient ex post equilibrium if there is an efficient ex post equilibrium of the English auction in the quasilinear* *setting where bidder i has value function*

$$v_i(s) = \hat{d}_i(s) \; \forall s \in S^n.$$

Thus, we can say that an English has an efficient ex post equilibrium if $\hat{d}_i(s)$ satisfies Krishna's (2003) average or weighted crossing conditions or the generalized single crossing condition of Birulin and Izmalkov (2011). Bidder strategies in the efficient ex post equilibrium in the normal good setting are identical to bidder strategies in the corresponding quasilinear setting, because an efficient ex post equilibrium in the latter implies an efficient ex post equilibrium in the former.

Corollary 1 extends Milgrom and Weber's (1982) results on the efficiency of English auctions in a symmetric quasilinear setting to a symmetric non-quasilinear setting. Milgrom and Weber (1982) characterize an ex post equilibrium of an English auction when bidders are symmetric. The example below presents the analogous model in our normal good setting. This setting will also serve as a useful benchmark when we compare our results in the good setting to results in the procurement setting.

Example 1. (Symmetric Model)

Let $S_i = [\underline{s}, \overline{s}] \subset \mathbb{R} \; \forall i \in \{1, \ldots, n\}$, where there is a u such that

$$u^1(m, (s_i, s_{-i})) = u_i^1(m, (s_i, s_{-i})) \; \forall (s_i, s_{-i}) \in S^n, \; m \in \mathbb{R}, \; i \in \{1, \ldots, n\},$$

$$u_i^0(0, s) = 0 \; \forall s \in [\underline{s}, \overline{s}]^n,$$

and

$$u^1(m, (s_i, s_{-i})) > u^1(m, (s_j, s_{-j})) \iff s_i > s_j.$$

In addition, we assume u^1 is continuous and increasing bidder signals. Then there is a $v_i(s)$ such that $v_i(s) = \hat{d}_i(s) \; \forall s \in S^n$. Corollary 1 shows that the efficient ex post equilibrium of the English auction in the above setting is equivalent to the efficient ex post equilibrium described in Milgrom and Weber (1982) when bidders have values $v_i(s) = \hat{d}_i(s)$. The mechanism is efficient because the our assumptions imply that the bidder with the highest signal has the highest willingness to pay.

In a recent paper, Hu, Matthews, and Zou (2015) provide necessary conditions for the existence efficient ex post equilibria in the risky asset setting. In their setting, the winner of the auction wins the asset, and the asset's monetary returns are determined by the realization of a random variable. They show that the English auction has an efficient ex post equilibria exists if bidders have non-decreasing absolute risk aversion their willingness to pay for the asset satisfies Krishna's (2003) average crossing or cyclical crossing conditions. They derive bid behavior by showing that bidders can perfectly infer the signals of their rivals when they drop out of the auction. Corollary shows that we can also obtain their result by noting that the risky asset is a normal good if bidder's have non-decreasing absolute risk aversion. When bidders have decreasing absolute risk aversion, they demand a lower risk premium on the risky asset as their wealth increases. Hence, the risky asset is normal because as the bidder becomes richer, their willingness to pay for the risky asset increases.

Example 2. (Assets with Ensuing Risk — Hu, Matthews, and Zou (2015))

Consider an English auction for a risky asset. Bidder i maximizes her expected utility from money and she has decreasing absolute risk aversion. If bidder i wins the asset, the she is paid a dividend of $v_i(s) + z_i$. The term $v_i(\cdot)$ satisfies Krishna (2003) average crossing condition and z_i is a type-independent random variable. Then, the risky asset is a normal good. If a bidder becomes richer, declining absolute risk aversion implies that she is less risk averse, and thus is willing to pay more for the risky asset. Moreover, Hu, Matthews, and Zou (2015) show that $\hat{d}_i(s)$ then satisfies Krishna's average crossing condition.[13] Thus, Corollary 1 implies that the English auction has an efficient ex post equilibrium that is equivalent to the efficient ex post equilibrium in a quasilinear setting where bidder values are $v_i(s) = \hat{d}_i(s)$.

5 Efficient Procurement Auctions

In this section, we study a procurement setting where bidders have positive wealth effects and we obtain results that are distinct from the results in the good setting. Theorem 2 shows that a mechanism has an ex post efficient equilibrium only if the mechanism has an ex post efficient equilibrium in the quasilinear setting. Efficiency in the corresponding quasilinear setting is a necessary, but not sufficient condition for efficiency in the auction setting. We provide a stricter condition on bidder preferences that is necessary and sufficient for the existence of a mechanism with an ex post efficient equilibrium. We then show that in a symmetric setting there is no mechanism with an ex post efficient equilibrium if there is sufficiently strong interdependence between bidders.

A bidder's reservation cost in the procurement setting is the absolute value of the bidder's willingness to pay for the task,

$$c_i(s) = |\hat{d}_i(s)|.$$

A bidder who has received no upfront payment would accept a take-it-or-leave-it offer to

[13] If values satisfy Krishna's cyclical crossing condition, then bidders' willingness to pay also satisfies cyclical crossing.

complete the task if and only if she is offered an amount that exceeds her reservation cost.

We similarly define $a_i(m, s)$ as the analog to a bidder's willingness to sell — we call a_i bidder i's willingness to offer. It is the most amount of money that bidder i would be willing to offer someone to complete the procurement task for them, conditional on having been assigned the task and receiving m from the auctioneer,

$$a_i(m, s) = |h_i(m, s)|.$$

Thus, a bidder who is responsible for completing the procurement task, and has received m in compensation, would prefer to pay another bidder p to complete the task if and only if $p \leq a_i(m, s)$. Note that positive wealth effects imply that a_i is increasing in the first argument. Moreover, a_i is strictly increasing in the first argument if bidders have strictly positive wealth effects.

Below, we present two examples of procurement settings where bidders have positive wealth effects.

Example 3. (Additively separable case)

There are n bidders and each bidder i gets signal $S_i \in [0, 1]$. Bidder i gets disutility of $s_i + \alpha \sum_{j \neq i} s_j$ when she completes the procurement task. In addition, bidder i gets utility $g(m)$ when when receives $m \in \mathbb{R}$ in transfers, where g is strictly increasing, continuous, and strictly concave. It is without loss of generality to assume that bidder i has wealth zero and $g(0) = 0$.[14] Thus, bidder i has utility function

$$u_i^x(m, s) = -(s_i + \alpha \sum_{j \neq i} s_j)x + g(m).$$

Thus,

$$c_i(s) = g^{-1}(s_i + \alpha \sum_{j \neq i} s_j),$$

and

$$a_i(m, s) = g^{-1}((s_i + \alpha \sum_{j \neq i} s_j) + g(m)).$$

Note that a_i is strictly increasing in m because both g^{-1} and g are strictly increasing. That is, bidder i is willing to offer more money to have another complete the task on her behalf if we increase her initial wealth.

For the second case, we look at a procurement setting with limited liability that is studied

[14]We are concerned about deviations of initial wealth. We may suitably normalize g to be such that bidder i has initial wealth $w_i > 0$, or allow for binding budgets.

by Burguet, Ganuza, and Hauk (2012).

Example 4. (Procurement with limited liability — Burguet, Ganuza, and Hauk (2012))

Consider an auction to procure a government contract. There are 2 firms that compete to win the contract. The monetary cost of completing the project, $\kappa \in [0, 1]$, is unknown at the time the auction is conducted. Each firm has a privately known asset level, $s_i \in [0, 1]$ and has limited liability. If the project is assigned to firm i for p and $s_i + p < \kappa$, the project is unfinished and limited liability prevents the government from pursuing damages.

A bidder's utility is normalized to zero if she does not win and makes no transfers. In addition, the expected utility of bidder i after winning for a price of p is

$$u_i^1(p, s_i) = s_i + p - E[\kappa|\kappa \leq s_i + p].$$

Recall that $a_i(m, s)$ is defined as solving

$$u_i^1(m, s) = u_i^0(m - a_i, s),$$

or equivalently, $a_i(m, s)$

$$a_i(m, s) = \mathbb{E}[\kappa|\kappa \leq s_i + m] - s_i.$$

Since the expectation term is increasing in m, then bidder i's willingness to offer is also increasing in m, indicating that bidder i has positive wealth effects.

The last example presented is a pure private value environment. This is nested in our setting. We could also generalize the example to include interdependence by assuming that bidder beliefs on κ are affiliated with their signals.

Returning to the general procurement setting, Theorem 2 shows that if a mechanism Γ satisfies (1) IR, (2) no subsidies, (3) EPIC, and (4) efficiency, then the bidder with the lowest costs wins the task. Or equivalently, if Γ satisfies Properties (1)–(4) in the non-quasilinear setting then Γ satisfies Properties (1)–(4) in a corresponding quasilinear setting where bidder i has reservation cost $c_i(s)$. Note that Theorem 2 gives a necessary, but not sufficient condition for efficient implementation.

Theorem 2. *If Γ satisfies (1) IR, (2) no subsidies, (3) EPIC, and (4) efficiency, then*

$$q_i(s) = 1 \implies c_i(s) \leq \min_{j \neq i} c_j(s).$$

The proof is by contradiction. Suppose a mechanism satisfies Properties (1)–(4) and assigns the task bidder i and bidder i does not have the lowest reservation cost. If bidder i

wins the project and is paid an amount exactly equal to her reservation cost, then she would be willing to offer another bidder up to her reservation cost to avoid having to complete the project. That is,

$$c_i(s) = a_i(c_i(s), s),$$

where the above expression follows directly from the definitions of c_i and a_i. Yet, the mechanism satisfies IR, so the winning bidder is paid (weakly) more than her reservation cost to complete the task. Or equivalently, bidder i is paid her reservation cost and then given an additional non-negative refund. The refund makes bidder i wealthier. Since bidder i has strictly positive wealth effects, then the increase in her wealth weakly increases the amount she is willing to offer another bidder to complete the procurement task. Thus, she is willing to offer an another bidder an amount that is at least her reservation cost $c_i(s)$ to complete the procurement task for her. Yet if the winning bidder does not have the lowest reservation cost, then there is another bidder j who is willing to complete the task for an amount that is below bidder i's reservation cost. Hence, there is a Pareto improving trade where bidder i pays bidder j to complete the procurement task.

While Theorem 2 places necessary conditions on the allocation rule, it is not a sufficient condition for efficiency. Instead, we find necessary and sufficient conditions for efficient implementation by studying a class of candidate mechanisms. We show that there is a mechanism that satisfies (1) IR, (2) no subsidy, (3) EPIC, and (4) efficiency if and only if there is a candidate mechanism that satisfies (1)–(4).

A candidate mechanism Γ^* has an allocation rule that always assigns the task to a lowest cost bidder,

$$q_i(s) = 1 \implies c_i(s) \leq \max_{j \neq i} c_j(s), \text{ and } \sum_{i=1}^{n} q_i(s) = 1 \ \forall s = (s_1, \ldots, s_n) \in S^n.$$

In addition, in a candidate mechanism Γ^* the payment rule states that if bidder i wins, then she is paid $\omega_i^*(s_{-i})$ where

$$\omega_i^*(s_{-i}) = \sup_{\tilde{s}_i \in S_i} c_i(\tilde{s}_i, s_{-i}) \text{ s.t. } q_i(\tilde{s}_i, s_{-i}) = 1.$$

Note that $\omega_i^*(s_{-i})$ is the highest possible reservation cost that bidder i could have reported, under the constraint that bidder i is still selected as the winner. The outcome under any candidate mechanism is unique when there is no tie for the lowest cost bidder.

Proposition 2. *There exists a mechanism that satisfies (1) IR, (2) EPIC, (3) no subsidy, and (4) efficiency if and only if there is a candidate mechanism Γ^* that satisfies (1)–(4).*

The proof can be understood intuitively. First we show that if $\tilde{\Gamma}$ satisfies EPIC, then so does a candidate mechanism Γ^*. We use Theorem 2 and Proposition 1 to show that there is a candidate mechanism Γ^* with the same allocation rule as mechanism $\tilde{\Gamma}$. We use this construction to show that Γ^* satisfies EPIC if $\tilde{\Gamma}$ satisfies EPIC.

Then we show that the candidate mechanism Γ^* satisfies efficiency if mechanism $\tilde{\Gamma}$ does. If bidder i wins the task in mechanism $\tilde{\Gamma}$, then IR implies that she is paid an amount that (weakly) exceeds $\omega_i^*(s_{-i})$.[15] Thus, mechanism $\tilde{\Gamma}$ and Γ^* both have the same allocation rule, and the winning bidder is paid weakly more in mechanism $\tilde{\Gamma}$. Positive wealth effects then imply that the winning bidder is willing to offer more money to have one of her rivals to complete the procurement task in outcome implied by mechanism $\tilde{\Gamma}$ versus the outcome implied by mechanism Γ^*. Thus, if there are no ex post Pareto improving trades in mechanism $\tilde{\Gamma}$ — where the winner is willing to offer more money to have another bidder complete the procurement task, then there are no ex post Pareto improving trades under mechanism Γ^* — where the winning bidder is less willing to pay another bidder to complete the procurement task. Hence, there is a candidate mechanism Γ^* that satisfies Properties (1)–(4) when there exists a mechanism $\tilde{\Gamma}$ that satisfies Properties (1)–(4).

5.1 A Symmetric Procurement Environment

In this section, we show that there is no mechanism that satisfies (1)IR, (2) no subsidy, (3) EPIC, and (4) efficiency when there is sufficiently strong interdependence in bidder demands. We study a symmetric environment that is parametrized by α, which measures the level interdependence of bidder preferences. We show that the necessary and sufficient condition for efficient design from Proposition 2 is satisfied if and only if the level of interdependence α is sufficiently small.

We assume bidders have single dimensional types and $S_i = [0,1] \subset \mathbb{R}\ \forall i \in \{1,\dots,n\}$ and we normalize a bidder's utility to be zero if she does not win and does not make any payment,

$$u_i^0(0,s) = 0\ \forall s \in S^n.$$

In addition, there is function u such that

$$u_i^1(m,s) = u(m, s_i + \alpha \sum_{j \neq i} s_j),\ \forall i \in \{1,\dots,n\},$$

where u is continuous and strictly increasing in both arguments. We assume bidders have

[15] If bidder i is paid an amount p that is less than $\omega_i^*(s_{-i})$, then there exists a $\tilde{s}_i$ such that bidder i has the lowest costs, yet $p < c_i(\tilde{s}_i, s_{-i})$. This would violate IR.

strictly positive wealth effects. If $\alpha = 0$, this is a pure private value setting and if $\alpha = 1$ this is a pure common value setting. For notational simplicity, we write bidder i's reservation cost $c_i(s)$ as $c(s_i + \alpha \sum_{j \neq i} s_j)$. Note that $c_i(\cdot)$ is strictly decreasing.

Next, we derive conditions under which a candidate mechanism satisfies Properties (1)-(4). We show that the candidate mechanism violates efficiency when the level of interdependence among bidders α is sufficiently large. Hence, Proposition 2 implies that there is exists a mechanism that satisfies Properties (1)-(4) if and only if α is sufficiently small. This is stated in Theorem 3 below.

Theorem 3. *There exists and $\alpha^* \in [0, 1)$ such that there exists a mechanism that satisfies (1) IR, (2) no subsidies, (3) EPIC, and (4) efficiency if and only if $\alpha \leq \alpha^*$.*

The intuition for the result can be understood by studying the extreme cases of pure common values ($\alpha = 1$) and pure private values ($\alpha = 0$) with two bidders.

In the pure common value setting, suppose that bidder 1 wins the procurement contract under a candidate mechanism Γ^*. Because the mechanism is individually rational 1 is paid an amount that exceeds her reservation cost.[16] In addition, since we are in a pure common value setting, then bidder 2 also has the same reservation cost as bidder 1. If bidder 1 was paid exactly her reservation cost, then she would be willing to offer bidder 2 an amount up to her reservation cost to complete the task. This is because bidder 1 is indifferent between completing the task and not completing the task when she is paid her reservation cost. Yet, bidder 1 is typically paid more than her reservation cost because she is compensated for her information rents in a mechanism that satisfies EPIC. Thus, it is as though bidder 1 was paid her reservation cost, and then given some additional money. When bidder 1 is given additional money, she willing to offer bidder 2 more money to complete the task, because bidder 1 has strictly positive wealth effects. Thus, bidder 1 is willing to pay bidder 2 an amount that strictly exceeds her reservation cost to complete the project. In addition, bidder 2 is willing to complete the task for an amount that equals her reservation cost, $c(s_1 + s_2)$. Therefore, bidder 1 is willing to offer an amount that strictly exceeds bidder 2's reservation cost in order to have bidder 2 complete the project. Thus, there is an ex post Pareto improving trade where bidder 1 pays bidder 2 to complete the task, and mechanism Γ^* is not efficient. Hence, Proposition 2 implies that there is no efficient mechanism.

In contrast, when bidder's have pure private values ($\alpha = 0$), then mechanism Γ^* is efficient (and equivalent to a second price auction). We can see the intuition by again considering two bidder setting where bidder 1 wins the procurement contract under mechanism Γ^*. Thus, $s_1 > s_2$. When bidder 1 wins, she is paid her rival's reservation cost $c(s_2)$, where $c(s_2) > c(s_1)$.

[16] More formally, bidder 1 is paid $w_1^*(s_2) = c(2s_2) > c(s_1 + s_2)$ if $s_1 > s_2$.

Therefore, bidder 1 is paid an amount that exceeds her reservation cost, and her utility increases.[17] If bidder 1 wanted to pay bidder 2 to complete the task, then bidder 1 must offer bidder 2 at least $c(s_2)$. This is because $c(s_2)$ is bidder $2's$ reservation cost. Yet, if bidder 1 pays bidder 2 an amount $p \geq c(s_2)$ to complete the task, she does not have to complete the task and her wealth decreases (weakly) relative to her wealth prior to the auction. Bidder 1's wealth decreases because she pays bidder 2 an amount p that exceeds her the amount the government paid her upon winning $c(s_2)$. Thus, if bidder 1 pay bidder 2 an amount that she would be willing to accept, then she (bidder 1) is made worse off. Hence, there are no Pareto improving trades between the two bidders and the candidate mechanism Γ^* is efficient.

Note that in our symmetric setting, a candidate mechanism Γ^* can be implemented via the English auction. Thus, we can say that there is an efficient auction in our symmetric procurement if and only if the English auction is efficient.

Corollary 2. *The English auction has a symmetric ex post equilibrium that is equivalent to the outcome of a candidate mechanism* Γ^*.

Corollary 2 follows from Theorem 10 in Milgrom and Weber (1982). Milgrom and Weber (1982) show that in a symmetric quasilinear setting, the English auction has an ex post equilibrium where the high value bidder (here lowest cost) wins the object. Proposition 1 then implies that the English auction satisfies IR, no subsidies, and EPIC in non-quasilinear setting because it satisfies the three conditions in the corresponding quasilinear setting.

6 Conclusion

We study efficient and ex post implementable mechanisms when bidders have non-quasilinear preferences with positive wealth effects. Our setting nests well-studied cases such as auctions with risk averse bidders, auctions with financially constrained bidders, and auctions for risky assets. We study the incentive and efficiency properties of mechanisms in the non-quasilinear setting by considering the mechanism's corresponding quasilinear setting.

We show that when the object being allocated is a good, a mechanism is efficient in the non-quasilinear setting if the mechanism is efficient in the corresponding quasilinear setting. This implies that conditions guaranteeing the efficiency of the English auction in a quasilinear setting translate to the non-quasilinear setting when bidders have positive wealth effects. Moreover, this gives us a simple method for computing equilibrium bid behavior in English auctions without quasilinearity. We get distinct results in procurement settings.

[17] $c(s_2) > c(s_1) \implies u(1, c(s_2), s_1) > u(1, c(s_1), s_1) = u(0, 0, s_1) = 0$ where the first equality follows from the definition of $c(\cdot)$.

We show that there are cases where there is no mechanism that has desirable incentive and efficiency properties, even if there is an ex post implementable mechanism that assigns the task to the lowest cost bidder.

The methodology used in this paper could also be useful in further research on other mechanism design settings without quasilinearity. More precisely, we would be interested in studying how we could construct corresponding quasilinear settings in other mechanism design problems where agents have non-quasilinear preferences. We think that this could help us to understand the limits of efficient and ex-post implementation without quasilinearity in multi-unit auctions, combinatorial assignment problems, or double auction models.

7 Proofs

Proof of Lemma 1

Proof. First, we show that the three inequality conditions imply IR. If $\tau_i^0(s_{-i}) \leq 0$, then

$$q_i(s) = 0 \implies u_i^0(t_i(s), s) = u_i^0(-\tau_i^0(s_{-i}), s) \geq u_i^0(0, s),$$

and if $q_i(s) = 1$, then $d_i(-\tau_i^0(s_{-i}), s) \geq \tau_i^1(s_{-i})$ and

$$u_i^1(t_i(s), s) = u_i^1(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) \geq u_i^1(-\tau_i^0(s_{-i}) - d_i, s) = u_i^0(-\tau_i^0(s_{-i}), s) \geq u_i^0(0, s),$$

where $d_i = d_i(-\tau_i^0(s_{-i}), s)$. Thus, the three conditions imply Γ satisfies IR.

Second, we show that the three conditions imply that Γ satisfies EPIC. Let $s \in S$ and $s' \in S$ be such that $s_i \neq s_i'$ but $s_{-i} = s_{-i}'$. If $q_i(s) = q_i(s')$, then by construction $t_i(s) = t_i(s')$, and

$$u_i^{q_i(s)}(t_i(s), s) = u_i^{q_i(s')}(t_i(s'), s').$$

If $q_i(s) = 0$ and $q_i(s') = 1$, then

$$u_i^1(t_i(s'), s') = u_i^1(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s') \geq u_i^0(-\tau_i^0(s_{-i}), s') = u_i^0(t_i(s), s'),$$

where the inequality follows because $d_i(-\tau_i^0(s_{-i}), s') \geq \tau_i^1(s_{-i})$. Similarly,

$$u_i^0(t_i(s), s) = u_i^0(-\tau_i^0(s_{-i}), s) \geq u_i^1(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) = u_i^1(t_i(s'), s),$$

because $\tau_i^1(s_{-i}) \geq d_i(-\tau_i^0(s_{-i}), s)$. Thus, $\forall s \in S$ and any $s' \in S$ such that $s_i \neq s_i'$ but $s_{-i} = s_{-i}'$

$$u_i^{q_i(s)}(t_i(s), s) \geq u_i^{q_i(s')}(t_i(s'), s).$$

Hence, $\Gamma = (q, t)$ satisfies EPIC.

To complete the proof we show that if Γ satisfies EPIC and IR, then it satisfies the three inequality conditions. First, we show $\tau_i^0(s_{-i}) \leq 0$. If there exists an $s_i \in S_i$ such that $q_i(s_i, s_{-i}) = 0$, then IR implies $t_i(s_i, s_{-i}) \geq 0$ and by definition $\tau_i^0(s_{-i}) = -t_i(s_i, s_{-i}) \leq 0$. If $q_i(s_i, s_{-i}) = 1 \; \forall s_i \in S_i$, then $\tau_i^0(s_{-i}) = 0$ by construction. Thus $\tau_i^0(s_{-i}) \leq 0 \; \forall s_{-i}$.

Second, we show that if $q_i(s) = 0$, then $\tau_i^1(s_{-i}) \geq d_i(-\tau_i^0(s_{-i}), s)$. The proof is by contradiction. Suppose that Γ satisfies EPIC and IR, yet there is a $s \in S^n$ such that $q_i(s) = 0$ and $d_i(-\tau_i^0(s_{-i}), s) > \tau_i^1(s_{-i})$. Since $d_i(-\tau_i^0(s_{-i}), s)$ is finite, so is $\tau_i^1(s_{-i})$. Therefore, $\exists s_i'$ *s.t.* $q_i(s_i', s_{-i}) = 1$. Thus, we have a violation of EPIC because bidder i gets a strictly great payoff by reporting type s_i'. Thus, $q_i(s) = 0 \implies \tau_i^1(s_{-i}) \geq d_i(-\tau_i^0(s_{-i}), s)$. An identical argument proves the third condition, $q_i(s) = 1 \implies d_i(-\tau_i^0(s_{-i}), s) \geq \tau_i^1(s_{-i})$. □

Proof of Proposition 1.

Proof. If Γ satisfies EPIC and IR in the non-quasilinear setting, then Lemma 1 shows using $v_i(s) = d_i(-\tau_i^0(s_{-i}), s)$ that

$$q_i(s) = 0 \implies \tau_i^1(s_{-i}) \geq v_i(s),$$

and

$$q_i(s) = 1 \implies v_i(s) \geq \tau_i^1(s_{-i}).$$

Lemma 1 shows that the above two inequalities conditions are necessary and sufficient to show that Γ satisfies EPIC and IR in setting $QL(\Gamma)$.

An identical argument shows that if Γ satisfies EPIC and IR in setting $QL(\Gamma)$, then Γ satisfies EPIC and IR in the non-quasilinear setting. □

Proof of Theorem 1

Proof. If Γ satisfies EPIC and IR in setting $QL(\Gamma)$, then Proposition 1 shows that Γ satisfies EPIC and IR in the non-quasilinear setting. Thus, we need to show that if Γ satisfies EPIC, IR, and efficiency in setting $QL(\Gamma)$, then Γ satisfies efficiency in the non-quasilinear setting. We assume Γ is efficient in setting $QL(\Gamma)$. Thus,

$$v_i(s) \geq v_j(s) \text{ if } q_i(s) = 1 \text{ and } j \neq i.$$

By construction of v_i, then

$$d_i(-\tau_i^0(s_{-i}), s) = v_i(s) \geq v_j(s) = d_j(-\tau_j^0(s_{-j}), s).$$

In addition, EPIC implies that

$$d_i(-\tau_i^0(s_{-i}), s) \geq \tau_i^1(s_{-i}) \text{ if } q_i(s) = 1.$$

Thus, if we let $d^* = d_i(-\tau_i^0(s_{-i}), s)$, then

$$h_i(-\tau_i^0(s_{-i}) - d^*, s) = d_i(-\tau_i^0(s_{-i}), s) \implies h_i(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) \geq h_i(-\tau_i^0(s_{-i}) - d^*, s)$$

where the inequality holds because positive wealth effects imply that h_i is increasing in the second argument. Thus,

$$h_i(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) \geq d_i(-\tau_i^0(s_{-i}), s) \geq d_j(-\tau_j^0(s_{-j}), s) \text{ if } q_i(s) = 1.$$

Therefore, there are no ex post Pareto improving trades between the winning bidder and her rivals, and Remark 2 shows that this implies that Γ is efficient. □

Example 5

This example shows that efficiency in a corresponding quasilinear setting is not necessary for efficiency in our non-quasilinear setting. Thus, we can construct a mechanism that does not assign the unit to the bidder with the highest willingness to pay, yet is efficient.

Suppose there are two bidders and that $S_1 = S_2 = [1, 2]$, where

$$u_i^x(m, s) = \left(s_i + \frac{1}{2}s_j\right) x + ln(m + 1).$$

Then,

$$\hat{d}_i(s) = 1 - e^{-(s_i + \frac{1}{2}s_j)},$$

and

$$h_i(t, s) = (1 + t)(e^{s_i + \frac{1}{2}s_j} - 1).$$

Let Γ be a mechanism where

$$q_1(s) = 1 \; and, \; q_2(s) = t_1(s) = t_2(s) = 0 \; \forall s \in S^2.$$

The proposed mechanism always assigns the good to bidder 1 for free. The construction of the mechanism immediately implies that it satisfies (1) IR, (2) no subsidies, and (3) efficiency.

The mechanism is efficient if there are no ex post Pareto improving trades between the two bidders. This holds because

$$h_1(0, s) \geq \hat{d}_2(s) \; \forall s \in S^2$$

because the above condition is equivalent to

$$e^{s_1 + \frac{1}{2}s_2} - 1 > 1 - e^{-s_2 - \frac{1}{2}s_1},$$

$$e^{s_1 + \frac{1}{2}s_2} + e^{-s_2 - \frac{1}{2}s_1} > 2,$$

which holds because $e^{s_1+\frac{1}{2}s_2} > 2 \; \forall s \in S^2$.

Proof of Theorem 2

Proof. The proof is by contradiction. Suppose Γ satisfies (1)–(4) yet there is a $s \in S^n$ where $q_i(s) = 1$ and a bidder $j \neq i$ such that $c_j(s) < c_i(s)$. Since Γ satisfies IR, then we know that if bidder i wins the procurement auction, she is paid some amount p where $p \geq c_i(s)$. Moreover positive wealth effects imply that $a_i(p, s)$ is increasing in p, and the construction of a_i and c_i imply that

$$a_i(c_i(s), s) = c_i(s).$$

Thus,

$$p \geq c_i(s) \implies a_i(p, s) \geq c_i(s).$$

Thus, bidder i is weakly better off by paying another bidder $c_i(s)$ to complete the task. Bidder j is made strictly better off completing the task and being paid $c_i(s)$, because her reservation cost $c_j(s)$ is strictly smaller than $c_i(s)$ by assumption. Hence there is a Pareto improving trade where bidder i pays bidder j the amount $c_i(s)$ to complete the task. This contradicts efficiency. □

Proof of Proposition 2

If Γ^* is satisfies Properties (1)–(4), then obviously there exists a mechanism that satisfies (1)–(4).

To prove the opposite direction suppose that $\tilde{\Gamma}$ satisfies Properties (1)-(4). We show that this implies that there is a candidate mechanism Γ^* that satisfies Properties (1)-(4).

By construction any candidate mechanism satisfies IR and no subsidies.

Next, I show that there is a candidate mechanism satisfies EPIC. Consider the candidate mechanism Γ^* that has the same allocation rule as $\tilde{\Gamma}$. We know such a mechanism exists because Theorem 2 ensures that any mechanism that is efficient only sells the good to a lowest cost bidder. Let $\mathcal{S}_i(s_{-i}) = \{s_i | \tilde{q}_i(s_i, s_{-i}) = q_i^*(s_i, s_{-i}) = 1\}$.

By construction $-\tau_i^{1*}(s_{-i}) = \omega_i^*(s_{-i})$ and $\tau_i^{0*}(s_{-i}) = 0$ for mechanism Γ^*. Suppose that Γ^* violates EPIC. Then Lemma 1 implies that there is either a $s_i \in \mathcal{S}_i(s_{-i})$ such that

$$-\tau_i^{1*}(s_{-i}) = \omega_i^*(s_{-i}) < c_i(s_i, s_{-i}) = -\hat{d}_i(s_i, s_{-i}),$$

or there is a $s_i \notin \mathcal{S}_i(s_{-i})$ such that

$$-\hat{d}_i(s_i, s_{-i}) = c_i(s_i, s_{-i}) > \omega_i^*(s_{-i}) = -\tau_i^{1*}(s_{-i}).$$

Yet, the first can not hold the construction of $\omega_i^*(s_{-i})$ implies that

$$\omega_i^*(s_{-i}) \geq c_i(s_i, s_{-i}) \; \forall s_i \in \mathcal{S}_i(s_{-i}).$$

Thus, if Γ^* violates EPIC, then there exists a $s_i \notin \mathcal{S}_i(s_{-i})$ such that,

$$c_i(s_i, s_{-i}) < \omega_i^*(s_{-i}).$$

Lemma 1 implies that since $\tilde{\Gamma}$ satisfies EPIC, then

$$s_i \notin \mathcal{S}_i(s_{-i}) \implies -\tilde{\tau_i^1}(s_{-i}) \leq c_i(s_i, s_{-i}).$$

Thus, if Γ^* violates EPIC, then there exists a $s_i \notin \mathcal{S}_i(s_{-i})$, such that

$$-\tilde{\tau_i^1}(s_{-i}) < \omega_i^*(s_{-i}).$$

Yet then there exists a $s_i \in \mathcal{S}_i(s_{-i})$ such that

$$-\tilde{\tau_i^1}(s_{-i}) < c_i(s_i, s_{-i})$$

because the $\omega_i^*(s_{-i})$ is defined as $\omega_i^*(s_{-i}) = \sup_{\tilde{s}_i \in \mathcal{S}_i(s_{-i})} c_i(\tilde{s}_i, s_{-i})$. The above equation and Lemma 1 then implies that mechanism $\tilde{\Gamma}$ violates EPIC, which contradicts our assumption that $\tilde{\Gamma}$ satisfies EPIC. Hence, when $\tilde{\Gamma}$ satisfies Properties (1)-(4), then Γ^* satisfies EPIC.

Finally, we show that mechanism Γ^* is efficient. The above argument showed that

$$\omega_i^*(s_{-i}) \leq -\tilde{\tau}_i^1(s_{-i}) \; \forall s_{-i}.$$

In addition, since $\tilde{\Gamma}$ is efficient, there are no ex post Pareto improving trades among bidders,

$$s_i \in \mathcal{S}_i(s_{-i}) \implies \max_{j \neq i} c_j(s_i, s_{-i}) \geq a_i(-\tilde{\tau}_i^1(s_{-i}), (s_i, s_{-i})).$$

And positive wealth effects imply that $a_i(\tilde{\tau}_i^1(s_{-i}), s) \geq a_i(\omega_i^*(s_{-i}), s)$ since $\omega_i^*(s_{-i}) \leq -\tilde{\tau}_i^1(s_{-i}) \; \forall s_{-i}$. Thus,

$$s_i \in \mathcal{S}_i(s_{-i}) \implies \max_{j \neq i} c_j(s_i, s_{-i}) \geq a_i(\tilde{\omega}_i(s_{-i}), (s_i, s_{-i})) \geq a_i(\omega_i^*(s_{-i}), (s_i, s_{-i})),$$

and there are no ex post Pareto improving trades in mechanism Γ^* and mechanism Γ^* is efficient.

Proof of Theorem 3

Proof. By construction Γ^* satisfies Properties IR and no subsides. In addition, Γ^* satisfies EPIC because Proposition 1 states that mechanism Γ^* satisfies EPIC if and only if Γ^* satisfies EPIC in the corresponding quasilinear environment. The corresponding quasilinear setting is a subset of the preferences studied by Milgrom and Weber (1982), and the outcome of Γ^* is equivalent to the (ex post) implementable English auction.

A candidate mechanism Γ^* satisfies Properties (1)–(4) when $\alpha = 0$ because in the private value setting a candidate mechanism Γ^* is equivalent to the Vickrey allocation rule studied by Saitoh and Serizawa (2008). Saitoh and Serizawa show that the Vickrey allocation rule is efficient in the single unit setting, hence any mechanism Γ^* satisfies efficiency when $\alpha = 0$.

Then, we show that a candidate mechanism Γ^* is inefficient when $\alpha = 1$. Hence $c_i(s) = c_j(s)\ \forall s \in S^n$. Let $\mathcal{S}_i(s_{-i}) := \{s_i | q_i(s_i, s_{-i}) = 1\}$. Since $\sum_{i=1}^n q_i(s) = 1\ \forall s \in S^n$, then there exists a bidder i such that $s_i^h, s_i^\ell \in \mathcal{S}_i(s_{-i})$, where $s_i^h > s_i^\ell$. Then $\omega_i^*(s_{-i}) \geq c_i(s_i^\ell, s_{-i}) > c_i(s_i^h, s_{-i})$. Positive wealth effects imply that if $s^h = (s_i^h, s_{-i})$, then

$$c_j(s^h) = c_i(s^h) = a_i(c_i(s^h), s^h) < a_i(\omega_i^*(s_{-i}), s^h),$$

where the final inequality follows because bidder i has strictly positive wealth effects and $\omega_i^*(s_{-i}) > c_i(s^h)$. Thus, there is a Pareto improving trade between bidder i and bidder j. Hence, the candidate mechanism is inefficient. Thus, Proposition 2 implies that there is no mechanism that satisfies Properties (1)-(4) when $\alpha = 1$.

Finally, we show if there is a candidate mechanism Γ^* that satisfies efficiency in an environment with level of interdependence $\alpha^h < 1$, then there is a candidate mechanism that satisfies efficiency in any environment where the level of interdependence is $0 \leq \alpha^\ell < \alpha^h$. Fix $(s_1, \ldots, s_n) \in S^n$ and suppose without loss of generality that $s_1 \geq s_2 \geq s_j\ \forall j \neq 1, 2$. Let

$$\tilde{s}_1 = s_1 + s_2(\alpha^\ell - \alpha^h \tfrac{1+\alpha^\ell}{1+\alpha^h}),$$
$$\tilde{s}_2 = \tfrac{1+\alpha^\ell}{1+\alpha^h} s_2,$$
$$\tilde{s}_j = \tfrac{\alpha^\ell}{\alpha^h} s_j.$$

The construction of $(\tilde{s}_1, \ldots, \tilde{s}_n)$ is such that $\tilde{s}_j \in S$ and $\tilde{s}_j \leq s_j\ \forall j = 1, \ldots, n$. Both follow immediately from the construction of $\tilde{s}_j$ when $j = 2, \ldots, n$. When $j = 1$, note that $-1 < \alpha^\ell - \alpha^h \frac{1+\alpha^\ell}{1+\alpha^h} < 0$ because $\alpha^\ell < \alpha^h$. Hence, $s_2 < s_1 \implies 0 \leq \tilde{s}_1 \leq s_1$. Thus, $\tilde{s}_1 \in S$ and $\tilde{s}_1 \leq s_1$.

Moreover, $\tilde{s}_1 \geq \tilde{s}_2 \geq \tilde{s}_j$, where $j \neq 1, 2$. To prove the first inequality, note that if $s_1 = s_2$, then $\tilde{s}_1 = \tilde{s}_2$. In addition, $\tilde{s}_1$ is increasing in s_1 and $\tilde{s}_2$ is unchanged in s_1. Hence, $s_1 \geq s_2 \implies \tilde{s}_1 \geq \tilde{s}_2$. To prove the second inequality note that $1 \geq \alpha^h > \alpha^\ell \geq 0$. Hence

$\frac{1+\alpha^\ell}{1+\alpha^h} \geq \frac{\alpha^\ell}{1+\alpha^h}$, and thus

$$s_2 \geq s_j \forall j \neq 1,2 \implies \tilde{s}_2 = \frac{1+\alpha^\ell}{1+\alpha^h} s_2 \geq \frac{\alpha^\ell}{\alpha^h} s_j = \tilde{s}_j \ \forall j \neq 1,2.$$

Since $(\tilde{s}_1, \ldots, \tilde{s}_n) \in S^n$, then when the level of interdependence is α^h and $q_1(\tilde{s}) = 1$, we have that

$$c_2\left(\tilde{s}_2 + \alpha^h \sum_{j\neq 2} \tilde{s}_j\right) - a\left(c\left(\tilde{s}_2 + \alpha^h \sum_{j\neq 1} \tilde{s}_j\right), \tilde{s}_1 + \alpha^h \sum_{j\neq 1} \tilde{s}_j\right) \geq 0.$$

That is, the reservation cost of the lowest cost bidder of bidder 1's rivals exceeds bidder 1's willingness to offer another bidder to complete the task. This inequality holds because we have assume the candidate mechanism is efficient when the level of interdependence is α^h. Then, by our definition of $(\tilde{s}_1, \ldots, \tilde{s}^n)$,

$$\begin{aligned} s_2 + \alpha^\ell \textstyle\sum_{j\neq 2} s_j &= \tilde{s}_2 + \alpha^h \textstyle\sum_{j\neq 2} \tilde{s}_j, \\ s_1 + \alpha^\ell \textstyle\sum_{j\neq 1} s_j &= \tilde{s}_1 + \alpha^h \textstyle\sum_{j\neq 1} \tilde{s}_j, \\ s_2 + \alpha^\ell \textstyle\sum_{j\neq 1} s_j &= \tilde{s}_2 + \alpha^h \textstyle\sum_{j\neq 1} \tilde{s}_j. \end{aligned}$$

Hence

$$c_2\left(s_2 + \alpha^\ell \sum_{j\neq 2} s_j\right) - a\left(c\left(s_2 + \alpha^\ell \sum_{j\neq 1} s_j\right), s_1 + \alpha^\ell \sum_{j\neq 1} s_j\right) \geq 0.$$

Since this holds for arbitrary $(s_1, \ldots, s_n)$ such that $s_1 \geq s_2 \geq s_j \forall j \neq 1,2$, then we see that the reservation cost of the lowest cost losing bidder exceeds the willingness to offer of the winning bidder. Hence, a candidate mechanism (with the same allocation rule) is efficient when the level of interdependence is $\alpha^\ell < \alpha^h$.

Thus, there is an $\alpha^* \in [0,1)$ such that Γ^* is efficient if and only if $\alpha \leq \alpha^*$. $\square$

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