Uniform Price Auctions with a Last Accepted Bid Pricing Rule

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March 21, 2018

Abstract

We model multi-unit auctions in which bidders' valuations are multidimensional private information. We show that the *last accepted bid* uniform-pricing rule admits a unique equilibrium with a simple characterization; in comparison, the commonly-studied *first rejected bid* uniform-pricing rule admits many equilibria, many of which provide zero expected revenue. In a natural example, equilibrium strategies in the last accepted bid auction are constructed from familiar strategies for single-unit first price auctions. In contrast with the information pooling we prove to arise in the first rejected bid and *pay as bid* auctions, the unique equilibrium of the last accepted bid auction is fully revealing.

1 Introduction

In a uniform price auction, bidders submit demand functions to a seller who awards m homogeneous units to the highest m bids at a single clearing price. This per-unit price may be the last accepted bid, the first rejected bid, or any intermediate amount.[1] These rules govern well-known large scale auctions, such as those run by the U.S. Treasury and the independent system operators in charge of electricity distribution. They are also used to model decentralized markets under the guise of competition in supply functions [Klemperer and Meyer, 1989, Vives, 2011].

Among common multi-unit auction formats, including pay-as-bid and Vickrey pricing, the uniform price rule is especially attractive because it awards homogeneous units at a homogeneous price.[2] The uniform price auction is fair, in the sense that bidders never pay less that other bidders for the same number of units won. If a goal of the auction is price discovery, the uniform price rule

We gratefully acknowledge discussions and feedback from Larry Ausubel, Peter Cramton, Pino Lopomo, David McAdams, and Marek Pycia, as well as seminar audiences at INFORMS, Duke IO Theory, SEA, and Collegio Carlo Alberto.

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[1] These rules are easily modified to accommodate bidders submitting supply curves to sell goods, sale of a divisible good and/or a seller using a non-constant supply (or demand) schedule.

[2] In a pay-as-bid auction, the highest m bids are awarded goods each winning bidder must pay the sum of their winning bids. By Vickrey pricing, we mean charging a bidder who wins k units the sum of the k highest rejected bids.

is a natural choice. As indicated by models of decentralized markets, this pricing rule also aligns well with textbook descriptions of markets (see, e.g., Kyle [1989], Vives [2011]).

However, a number of studies emphasize unappealing properties of the equilibria that may arise in these auctions. First, when bidders have private information, their bidding problem can be complex. A bid for any unit — except possibly for the first unit — may determine the clearing price and hence influence the total amount paid. Consequently, bidders strategically reduce their bids below their demand, complicating inferences about the distribution of opposing bids when bidders have private information [Vickrey, 1961, Ausubel et al., 2014].[3] Second, and perhaps most concerning for practitioners, there may be equilibria that generate little to no revenue for the seller and reveal no private information [Back and Zender, 1993]. In these low-revenue equilibria bidders tacitly agree on a division of the goods and submit demand curves that implement this division at the lowest possible price. The third closely related issue is that there may be multiple equilibria, each with different allocations. Finally, while certainly true of many of the low-revenue equilibria, it has been argued that the equilibria of these auctions are generally inefficient [Ausubel et al., 2014].[4]

In this paper, we show that incentives in uniform price auctions — and the undesirable properties they imply — are closely tied to the assumption that bidders pay the first rejected bid, and equilibrium predictions are dramatically affected if a different clearing price is selected. In practice, market clearing prices are computed as the price which would clear a Walrasian market, taking reported bid curves to be actual demand curves. With discrete goods this typically leaves some discretion to the seller: when m units are available, markets will (weakly) clear as long as the price is between the m^{th} and $m+1^{\text{th}}$ highest bids.[5] Existing literature has tended to focus on the selection of the lowest market clearing price, which we term the *first rejected bid*. We focus instead on the *last accepted bid* pricing rule, in which the highest market clearing price is selected.[6]

[3]The degree of demand reduction for a unit may depend on the number of prior units bid on and the bidder's marginal values for those units, and importantly may differ across units. Intuitively, a bidder reduces his bid below his demand curve for the same reason a monopolist's marginal revenue curve falls below its demand curve.

In environments with private information, these facts complicate each competing bidder's inferences about the probability distribution of opposing bids. To avoid these difficulties, authors have restricted attention to divisible-good models in which bidders have linear demands determined by normally distributed intercepts [Kyle, 1989, Vives, 2011] or to cases in which bidders demand exactly two goods [Ausubel et al., 2014, Engelbrecht-Wiggans and Kahn, 1998]. With a pricing rule that specifies that the clearing price is equal to the first rejected bid, bidders have a weakly dominant strategy to bid their marginal value on the first unit. When two units are demanded, this gives a natural class of potential equilibria in which only one bid per bidder must be determined.

[4]Furthermore, the inefficiencies are not beneficial to the seller. In a single-unit auction inefficiency arises from non-allocation of the good, and the seller can use this as a threat to increase submitted bids. In a multi-unit auction inefficiency can arises from mis-allocation of the good, and a reassignment of goods that shifts quantities from a low-value bidder to a high-value bidder can improve surplus on both sides of the market.

[5]These two prices weakly clear the market according to reported demand: selecting the m^{th} highest bid as the market clearing price leaves this bidder indifferent between buying the good and not, while selecting the $m+1^{\text{th}}$ highest bid leaves this bidder indifferent between buying the good and not. Between these two prices are a continuum of prices which strictly clear the market.

[6]In a procurement auction these roles are reversed: the first rejected bid is the highest market clearing price and the last accepted bid is the lowest market clearing price. Our explicit focus in this paper is on sales auctions, but we use the notions of last accepted and first rejected bids to capture the generality of the arguments without regard to the different meaning of "high" and "low" bids in sales and procurement contexts.

This apparently small change in the rules of the auction re-aligns bidding incentives, leading to dramatic effects on equilibrium behavior. In a first rejected bid auction the price is set by the bid for a supra-marginal unit, while in a last accepted bid auction the price is set by a bid for an exactly marginal unit. Although the possibility of multiple pricing rules has been noticed by theoretical literature (see, e.g., McAdams [2003]), a natural reading of the uniform price auctions literature suggests that the difference in price selection rules has been ignored for two reasons. First, uniform price auctions are used as models of behavior in decentralized markets. Second, when the number of units, m, is large, there is not a meaningful distinction between selecting the m^{th} highest bid and the $m+1^{\text{th}}$ highest bid.[7] We show that the implicit assumption that the exact location of the market clearing price is unimportant ignores the different incentives induced by selection of one price or another.

We model multi-unit auctions with a novel private information framework in which bidders have multidimensional private information about their demand for any number of units of an indivisible good.[8] The model allows the flexible specification of each bidder's expected number of units demanded at each price, referred to as a bidder's mean demand curve, while imposing restrictions on the distribution of realized demand curves about this mean demand curve. We show that this model, applied to the last accepted bid uniform-pricing rule, is tractable and yields equilibrium behavior with several desirable properties, especially when compared to existing models.

We first show that with two bidders the equilibrium bids for *each* marginal unit take the form of bids from an asymmetric first price auction for a *single* unit. If the two bidders' demands are symmetric, the bid curves can be solved in closed form as in the first price auction. Whereas the modern literature on uniform price auctions — and multi-unit auctions more generally — emphasizes differences between equilibrium bidding strategies in multi-unit auctions and single-unit auctions, we find a close connection between our model of the uniform price auction and the standard model of the first price auction.[9] An immediate implication is that many of the results from the first price auction literature translate directly to our environment. For example, we are able to draw a connection between the relationship between two bidders' mean demand curves and how aggressively they bid in the auction. By extending the work of Maskin and Riley [2000] we can classify the mean demand curves as "strong" or "weak" and translate these demand curves to bidding aggressiveness.

While our theoretical results are relatively general, our precise analytical results rely on an algebraic assumption on demand which we term *market balance*. In a balanced market with n

[7] These two points intersect in the analysis of divisible-good models: when submitted bids are continuous demand curves, there is no distinction between the highest and lowest prices clearing the market.

[8] Previous work, including McAdams [2007b] and Reny [2011], provides equilibrium existence and some worked examples of multi-unit uniform price auctions with multidimensional types. To the best of our knowledge ours is the first analytical characterization with demands exceeding two units.

[9] Historically, the literature on multi-unit auctions has sometimes assumed (without analysis) they were strategically analogous to single-unit auctions; see, e.g., Friedman [1991]. Our results should not be mistaken for a return to this view. As stated above, our results suggest a strategic connection between single-unit first price auctions and multi-unit last accepted bid auctions.

bidders, each collection of $n-1$ bidders exactly demands the m units available for auction.[10] The intuitive mapping to a first price auction is clean in a balanced market. When a bidder's bid for unit k is relevant, it sets the market clearing price; then she has outbid exactly k of her opponents' bids.[11] When participants' values are given by ordered draws from a common distribution, this means that the bidder has defeated k draws from a common distribution. Conditional on just beating these k draws the bidder pays her bid for this unit, generating exactly the same incentives as the first price auction.

First price bidding equilibria satisfy two important properties that are not typically satisfied in multi-unit auctions: they are unique, and they are separating in the sense that values can be inferred from bids. Both properties are valuable for empirical or computational work. It is immediate that the two-bidder equilibrium we identify is separating. Further, we use techniques from the first price auction literature to prove that this equilibrium is unique in the space of separating strategies. From this point existing results from single-unit auctions can be applied to obtain numerical solutions to equilibrium strategies.[12] With more than two bidders, equilibrium strategies in the last accepted bid uniform price auction can no longer be directly identified with a corresponding first price equilibrium, but we show that the properties of uniqueness and separation continue to hold with more than two bidders.[13]

We compare the last accepted bid auction to the commonly-analyzed first rejected bid uniform price and the pay as bid rules. We extend known inefficiency results in both the first rejected bid and pay as bid auctions (see, e.g., Ausubel et al. [2014]) to our model of bidder values. This provides a clear contrast with the unique separating equilibrium in the last accepted bid auction with symmetric distributions: this equilibrium is always efficient, while neither the first rejected bid nor the pay as bid auctions are efficient. We expose a connection between this inefficiency and the tractability problems which are known to complicate the first rejected bid and pay as bid auctions. In these two auctions, we show that information is pooled in all well-behaved equilibria. It is therefore impossible to achieve efficiency, and solving analytically for equilibrium involves the determination of "pooled intervals" in the underlying type space, which imply regions over which the first order conditions cannot be naïvely applied.[14] This provides a clear contrast with the tractable unique equilibrium we find in the last accepted bid auction. Last, we show that the low-revenue ("collusive-seeming") equilibria which are known to plague the first rejected bid auction

[10] We thank Curt Taylor for pointing out that this need only hold when (potentially stochastic) demand is at its maximum.

[11] A bid which does not set the market clearing price is irrelevant in the sense that, from any ex post outcome, a slight increase or decrease would not affect utility.

[12] While equilibrium first order conditions can be compactly speficied in common multi-unit auction models, equilibrium computation remains an open problem.

[13] Formally, we show that the last accepted bid auction admits a unique separating equilibrium. This is in stark contrast to the first rejected bid and pay as bid auctions, which we prove admit no separating equilibria.

[14] In the first rejected bid auction, pooling arises for relatively low valuations for which the marginal gain associated with an increase in winning probability is outweighed by the increased probability of setting the market price. In the pay as bid auction, pooling arises due to the constraint that bids be weakly decreasing while agents would sometimes prefer to submit nonmonotone (or even weakly increasing) bid functions. Although the incentives underlying pooling behavior are distinct in these two auctions, they imply similar issues for tractability.

cannot be supported in the last accepted bid auction. These points together provide straightforward justification for employing the last accepted bid pricing rule.

Although it may be true that for a fixed set of strategies the bidders' payoffs are not affected much by choosing the last accepted bid or the first rejected bid as the clearing price, our results show that the choice of clearing price does have a significant effect on equilibrium strategies. The underlying reason is that best response bids are determined by conditioning on the low probability event that that particular bid is selected as the clearing price, and focusing on this event, our analysis makes clear that whether the clearing price is the last accepted bid or the first rejected bid has significant implications for how the bid is optimally chosen.[15] Roughly, in a first rejected bid auction a bid is payoff-relevant only if it is submitted for a unit which *is not* won, while in a last accepted bid auction a bid is payoff-relevant only if it is submitted for a unit which *is* won; the differing alignments of bidding incentives lead to drastically different bidding behavior. Since our model accommodates any number of units, the equilibrium in our model further provides a natural equilibrium selection in the divisible-good case, where the equilibrium need not be unique.

These results suggest the care be paid to the selection of salient features of equilibrium in auction models. For example, bidders report truthfully in the canonical equilibrium of a single-unit second price auction; it is known that the optimality of truthful reporting does not extend to multi-unit first-rejected bid pricing (see above, and also Back and Zender [1993], Engelbrecht-Wiggans and Kahn [1998], Wang and Zender [2002], and Ausubel et al. [2014] among many others).[16] It is also known that the intuitive behavior in a single-unit first price auction does not translate to multi-unit pay as bid auctions, in spite of bidders paying their bids in both settings (see, e.g., Woodward [2016]). Our results show strategic equivalence between single-unit first price auctions and multi-unit last accepted bid auctions, suggesting that the strategically salient feature of these auctions is the selection of the highest market clearing price. To our knowledge this has gone unaddressed in the literature.

1.1 Related literature

This paper builds on results from multi- and single-unit auctions, as well as divisible-good auctions. Results in multi-unit auctions are hampered by the intractability of the multi-dimensional analysis required to maximize bidder utility. Results that address this intractability typically manage to reduce the dimension of the bid space; successful efforts on this front include Vickrey [1961], Engelbrecht-Wiggans and Kahn [1998], Engelbrecht-Wiggans and Kahn [2002], Lebrun and Tremblay [2003], Bresky [2013], and Ausubel et al. [2014]. Faced with this difficulty in the multi-unit problem, a thread of literature has analyzed divisible-good models in which the quantity at auction is infinitely divisible. Continuity of equilibrium outcomes ensures that where predictions are continuous, large multi-unit auctions will be approximated by divisible-good models. Nonetheless

[15] This distinction is relevant until the number of units becomes so large that the environment approaches a divisible-good model, in which the distinction between last-accepted and first-rejected bids is (typically) inconsequential.

[16] In particular, the optimality of truthful reporting in a single-unit second price auction derives from its equivalence to a Vickrey auction. With multiple units this equivalence evaporates.

success in this approximation has been essentially limited to cases with constant marginal values (e.g., Wilson [1979], Back and Zender [1993], and Wang and Zender [2002]) or no private information (e.g., Holmberg [2009], Anderson et al. [2013], and Pycia and Woodward [2017]).[17] A related body of work looks at uniform price auctions as approximations of classical market behavior, where a single price dictates the payments to and from all participants (e.g., Kyle [1989], Klemperer and Meyer [1989], and Vives [2011]).

We emphasize the connection between the last accepted bid multi-unit and first price single-unit auctions, which allows us to leverage many results in first price auctions to quickly derive theoretical properties of our model. Early investigations of equilibrium uniqueness in first price include Plum [1992], Bajari [2001], and Maskin and Riley [2003], and these results have been extended by Lebrun [2006], Kirkegaard [2009], and Chawla and Hartline [2013]. among others. In our uniqueness results, we closely follow the approach of Lebrun [2006]. Structurally, our results on asymmetric bidders are in line with Lebrun [1999] and Maskin and Riley [2000]. In contrast with uniqueness, which derives from analogy to first price auctions, equilibrium existence is directly guaranteed by known theoretical results (McAdams [2003], McAdams [2006], Reny [2011], Woodward [2017], and others).

Our results continue the thread of research investigating how to improve mechanism outcomes without substantially altering the allocation mechanism. In single-unit auctions reserve prices can be used not only to induce more aggressive bidding behavior (Myerson [1981], and many others) but also to select away seller-pessimal equilibria (Graham and Marshall [1987], Lizzeri and Persico [2000], Blume and Heidhues [2004], Blume et al. [2009], Chassang and Ortner [2015]). In multi-unit auctions it is known that an elastic supply curve — a generalization of a reserve price — can improve the seller's revenue (LiCalzi and Pavan [2005], McAdams [2007a]). Our results show that price selection can be a valuable tool for selecting away undesirable equilibria, leaving the fundamentals of the auction essentially intact.[18]

Although we do not attempt an empirical exercise, our results are potentially useful for practitioners. The theoretical ambiguity of outcome rankings between pay as bid and uniform price auctions provides a natural opening for empirical work. Existing studies provide an ambiguous lesson in mechanism selection: depending on context, pay as bid outperforms uniform price (Fevrier et al. [2002], Kang and Puller [2008], and Marszalec [2017]), uniform price outperforms pay as bid (Armantier and Sbaï [2006], Castellanos and Oviedo [2008], and Armantier and Sbaï [2009]), there is no statistical difference between the two (Hortacsu and McAdams [2010]), and there is no practical difference between the two (Hortaçsu et al. [2018]). Notably, counterfactual estimates typically rely on bounding best response behavior, as in Hortacsu and McAdams [2010]. Our tractable model of bidder behavior may provide a method for making these predictions more precise.

[17] Empirical results, such as Häfner [2015], make further inroads by eliding the equilibrium computation question altogether: if agents are playing equilibrium strategies, they are directly observable and need not be imputed.

[18] To extend the single-unit analogy, first price auctions typically have unique equilibria while second price auctions typically have many equilibria, some of which admit zero revenue. In single-unit auctions price selection can be used as a tool for eliminating these zero-revenue equilibria.

The remainder of the paper is organized as follows. In Section 2, we analyze an example with two bidders and two goods to preview the main results. Section 3 introduces the general model. In Section 4, we characterize equilibrium in the last accepted bid uniform price auction, and in Section 5 we present results on the separability and uniqueness of these equilibria. Section 6 provides contrasting results for the first rejected bid uniform price auction and the pay as bid auction, and Section 7 concludes.

2 Leading example: two bidders and two goods

We begin with a simple example of our model. There are $n = 2$ bidders, $i \in \{1, 2\}$, each with demand for (up to) $m_i = 2$ units. An auctioneer is selling $m = 2$ units in a multi-unit auction: he solicits weakly decreasing demands for each unit, b^i, from each of the bidders, and awards the two units to the agent(s) submitting the two highest bids.

Bidders have independent private values: bidder i's value for her k^{th} unit is denoted v^i_k. For each bidder, v^i is determined by ordering two independent draws from a $\mathcal{U}(0, 1)$ distribution; in particular, v^i_k is (marginally) distributed according the k^{th} order statistic of a uniform distribution on $[0, 1]$, $v^i_k \sim \mathcal{U}_{(k)}(0, 1)$. Given a bid function b^i_k mapping bidder i's values to a bid for the k^{th} unit, denote the inverse bid function mapping bids to values by φ^i_k.[19]

Denote the marginal distribution of v^i_k by $F_{(k)}$. Because values are distributed as order statistics,

$$F_{(1)}(x) = x^2, \text{ and } F_{(2)}(x) = 2x - x^2.$$

We consider two payment rules. In *last accepted bid* (LAB), bidders pay the second-highest bid submitted for each unit they receive. In *first rejected bid* (FRB), bidders pay the third-highest bid submitted for each unit they receive.[20] We defer discussion of pay as bid auctions, in which bidders pay their submitted bid for each unit they receive, until later in the paper.[21]

2.1 Last accepted bid

In LAB, three statistical events are salient.[22] First, bidder i can win 2 units; this occurs when $b^i_2 \geq b^{-i}_1$, and the market-clearing price is b^i_2. Second, bidder i can win 1 unit while bidder $-i$ sets the price; this occurs when $b^i_1 \geq b^{-i}_1 > b^i_2$. Third, bidder i can win 1 unit and set the price; this occurs when $b^{-i}_1 > b^i_1 \geq b^{-i}_2$.

[19]In the analysis of this example we elide some technical details and focus on well-behaved strategies; in particular, bids are strictly increasing in value and are thus invertible, and tiebreaking is a probability-zero event, so there is no concern about allocations when bidders submit the same bid.

[20]As we will discuss later, LAB and FRB correspond to the highest and lowest (respectively) market-clearing prices in a Walrasian market with inelastic supply and demands given by the submitted bids.

[21]While the LAB and FRB uniform price auctions have clean analytical characterizations in this model, the informational pooling properties of the pay as bid auction — examined in detail in Section 6 — lead to intractability even in this relatively simple case with demand for only two units.

[22]There is also a fourth relevant event, that bidder i wins zero units. Because this yields 0 utility, it is of no consequence to the formal analysis.

In terms of these these events, interim utility in LAB can be expressed as

$$\begin{aligned} u^i\left(b^i;v^i\right) = &\left(v_1^i + v_2^i - 2b_2^i\right)\Pr\left(b_2^i \geq b_1^{-i}\right) \\ &+ \left(v_1^i - \mathbb{E}\left[b_1^{-i}\middle| b_1^i \geq b_1^{-i} > b_2^i\right]\right)\Pr\left(b_1^i \geq b_1^{-i} > b_2^i\right) \\ &+ \left(v_1^i - b_1^i\right)\Pr\left(b_1^{-i} > b_1^i \geq b_2^{-i}\right). \end{aligned}$$

As we show in Appendix A, in a symmetric equilibrium the bidders' first order conditions are given by

$$2\left(\varphi_1(b) - b\right)\left(1 - \varphi_2(b)\right)d\varphi_2(b) - \left(2\varphi_2(b) - \varphi_2(b)^2 - \varphi_1(b)^2\right) = 0; \qquad \text{(unit 1)}$$

$$\left(\varphi_2(b) - b\right)\varphi_1(b)\,d\varphi_1(b) - \varphi_1(b)^2 = 0. \qquad \text{(unit 2)}$$

These equations imply that in equilibrium,

$$\varphi_1(b) = \varphi_2(b) = 2b, \quad \text{and} \quad b_1(v) = b_2(v) = \frac{1}{2}v.$$

That is, equilibrium in LAB is exactly equilibrium in a standard first price auction for a single unit. It is immediate that this equilibrium is efficient.

2.2 First rejected bid

In FRB, three statistical events are salient. First, bidder i can win 2 units; this occurs when $b_2^i \geq b_1^{-i}$, and the market-clearing price is b_{-i}^1. Second, bidder i can win 1 unit and set the price; this occurs when $b_1^{-i} > b_2^i \geq b_2^{-i}$. Third, bidder i can win 1 unit while bidder $-i$ sets the price; this occurs when $b_1^i \geq b_2^{-i} \geq b_2^i$.

With these events, interim utility in FRB can be expressed as

$$\begin{aligned} u^i\left(b^i;v^i\right) = &\left(v_1^i + v_2^i - 2\mathbb{E}\left[b_1^{-i}\middle| b_2^i \geq b_1^{-i}\right]\right)\Pr\left(b_2^i \geq b_1^{-i}\right) \\ &+ \left(v_1^i - b_2^i\right)\Pr\left(b_1^{-i} > b_2^i \geq b_2^{-i}\right) \\ &+ \left(v_1^i - \mathbb{E}\left[b_2^{-i}\middle| b_1^i \geq b_2^{-i} > b_2^i\right]\right)\Pr\left(b_1^i \geq b_2^{-i} > b_2^i\right). \end{aligned}$$

As we show in Appendix A, in a symmetric equilibrium the bidders' first order conditions are given by

$$\left(\varphi_1(b) - b\right)\left(2 - 2\varphi_2(b)\right)d\varphi_2(b) = 0; \qquad \text{(unit 1)}$$

$$2\left(\varphi_2(b) - b\right)\varphi_1(b)\,d\varphi_1(b) - \left(2\varphi_2(b) - \varphi_2(b)^2 - \varphi_1(b)^2\right) = 0. \qquad \text{(unit 2)}$$

The first order condition with respect to the bid for the first unit, b_1^i, confirms the intuiton that truthful reporting is a weakly dominant strategy. This follows from standard second price auction logic: the bid for the first unit never sets the clearing price (when the agent wins) so it is effectively

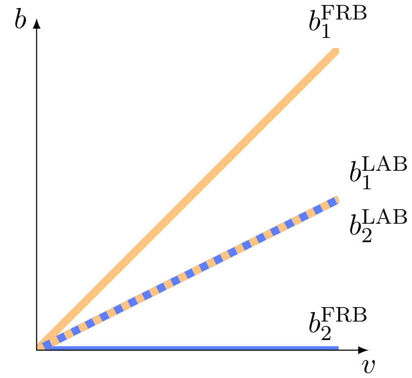

Figure 1: Equilibrium bids in the example LAB and FRB auctions in Section 2.

costless to increase the bid.[23]

In an equilibrium in which agents bid truthfully for their initial units the first order condition with respect to the second-unit bid is no longer (meaningfully) a differential equation: $\varphi_1(b) = b$, and hence $d\varphi_1(b) = 1$. Subtituting through, symmetric equilibirum bids for the second unit must solve

$$2\left(v_2 - b\right) b - \left(2v_2 - v_2^2 - b^2\right) = 0, \quad \text{or} \quad b = v_2 \pm \sqrt{2v_2^2 - 2v_2}. \tag{1}$$

Since $v_2 \in [0, 1]$, it must be that $2v_2^2 - 2v_2 \leq 0$; then the negative quadratic in (1) has no real zeroes, and the first order condition with respect to b_2 is negative everywhere. It follows that $b_2 = 0$ indentically, independent of v_2.

2.3 Comparison

Because bids in the LAB auction are independent of the unit for which they are submitted — that is, because $b_k^i(v) = v_k/2$ for both units — outcomes are efficient. This contrasts strongly with the results of the FRB auction. Because second-unit bids are always zero, inefficient outcomes will arise whenever one agent's value for her first unit is below the other agent's value for her second unit (in this example, the probability is 1/3).

Due to the linearity of bids in the LAB auction, expected revenues are given by half of the expected second-highest draw from four draws of a uniform distribution. Aggregate expected revenue is then 3/5, and per-unit expected revenue is 3/10. This again contrasts strongly with the expected revenue of the FRB auction. Because second-unit bids are always zero, the clearing price is always zero. Then aggregate and per-unit expected revenues are zero.

In the remainder of this paper we demonstrate that certain of these properties generalize. When market demand satisfies a simple algebraic condition ("market balance") there is an equilibrium of the LAB auction in which bids are independent of the unit for which they are submitted,

[23] Our technical results show that information confounding in equilibrium is independent of whether first-unit bids are truthful; it follows immediately that FRB equilibria are never efficient (except potentially in mathematically pathological cases). We conjecture that revenue-dominance of LAB over FRB holds across all FRB equilibria, but we do not formally demonstrate this result.

implying efficiency. We prove that there is a unique well-behaved[24] equilibrium in the LAB auction, establishing efficiency of a natural outcome of the auction. Contrariwise, we demonstrate that there is always information confounding and some degree of pooling in the FRB auction, thus all outcomes of the FRB auction are inefficient. We also demonstrate that the FRB auction always admits an equilibrium with zero expected revenue, while the LAB auction admits no such equilibrium.

3 Model

An auctioneer sells m units of a homogeneous good to n risk-neutral bidders who, with probability 1, have strictly positive aggregate demand for at least m units.[25] Bidder i values units according to the ordered realizations of m_i independent draws from the absolutely continuous distribution $F^i : [0, 1] \to [0, 1]$ with density f^i. The ordering ensures that marginal values are weakly declining for every realization. For example, the bidder's marginal value for the first unit is the first order statistic of m_i independent draws from F^i. When bidders are symmetric, $F^i = F$ for all i and some F. We denote the ordered vector of bidder i's valuations by v^i, so that v^i_k is her value for her k^{th} unit. By definition, $v^i_1 \geq \cdots \geq v^i_{m_i}$. For simplicity we sometimes reference the "$n \times \tilde{m}$ case", in which n symmetric agents each have demand for $m_i = \tilde{m}$ units. When $m = \sum_{j \neq i} m_j$ for any bidder i, we say that the market is *balanced.*

We consider sealed-bid auctions, where bidders submit weakly decreasing demand vectors (bid curves) to the auctioneer. Bidder i submits a weakly positive demand vector b^i, so that b^i_k is her bid for her k^{th} unit. Where helpful, we will take a mechanism design approach and consider bids as functions of bidders' private values, $b^i_k \equiv b^i_k(v^i)$. Without a reserve price, the auctioneer allocates the available units to the m highest bids.[26,27,28] Denote the maximum and minimum market-clearing prices by $\overline{p}$ and $\underline{p}$, respectively, where

$$\overline{p} = \min \left\{ p : \# \left\{ (i, k) : b^i_k \geq p \right\} \leq m \right\},$$
$$\underline{p} = \max \left\{ p : \# \left\{ (i, k) : b^i_k \geq p \right\} \geq m \right\}.$$

Each bidder is risk-neutral and her utility is quasilinear in payments. Conditional on allocation

[24] The proper notion of well-behavedness is defined later.

[25] We later consider the (stochastic) possibility of insufficient demand. This does not meaningfully complicate or change our results.

[26] Bid monotonicity is a constraint typically observed in practice. However, under the assumption that the auctioneer accepts bids in decreasing order bid monotonicity is also a simplifying assumption that can be made without loss of generality.

[27] Where the m^{th} highest bid is not well defined some form of tiebreaking or rationing is necessary. Because the tiebreaking rule is not of importance to our analysis, we leave it unspecified. This point has been noted in the multi-unit and divisible-good auction literature; see, e.g., Häfner [2015].

[28] When a nontrivial reserve price is present, bids are accepted in decreasing order until either all m units are allocated or there are no remaining bids weakly above the reserve price. For the most part our results are not meaningfully affected by the presence or absence of a reserve price (in light of what is known of behavior in single-unit auctions with reserve prices), so it is natural to ignore reserve price to avoid unnecessary technicalities.

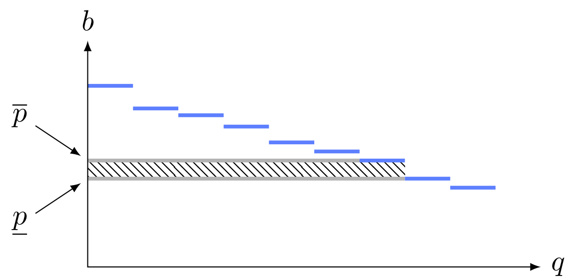

Figure 2: Maximum and minimum market clearing prices displayed on an aggregate demand curve, $D(q) = \inf\{p : \#\{(i,k) : b^i_k \geq p\} < q$, when $m = 7$ units are available.

q_i and payment t_i, bidder i's ex post utility is

$$u^i\left(q_i, t_i; v^i\right) = \left[\sum_{k=1}^{q_i} v^i_k\right] - t_i$$

We focus most of our attention on the *last accepted bid* (LAB) uniform-pricing rule, in which each bidder pays the same price for each unit she obtains, and this price is equal to the m^{th} highest bid; this is equivalent to clearing the market at $\overline{p}$, the highest market clearing price, and implies a transfer of $t_i^{\text{LAB}}(q_i) = \overline{p}q_i$. For mechanism comparisons, we also discuss the *first rejected bid* (FRB) uniform-pricing rule, where the per-unit price is the $(m+1)^{\text{th}}$ highest bid (equivalent to the lowest market clearing price) and the transfer is $t_i^{\text{FRB}}(q_i) = \underline{p}q_i$, and the *pay as bid* (PAB) pricing rule, in which for each unit a bidder receives she pays her bid for this specific unit, and the transfer is $t_i^{\text{PAB}}(q_i) = \sum_{k=1}^{q_i} b^i_k$.

Market clearing implies that bidder i receives unit k if and only if her opponents receive (in aggregate) less than $m - k + 1$ units.[29] It is helpful to consider bidder i competing for her k^{th} unit against the aggregate demand of her opponents for $m - k + 1$ units. Let H^i_{m-k+1} be the marginal distribution of her opponents' $m - k + 1^{\text{th}}$ highest bid, and let h^i_{m-k+1} be the associated density (where well-defined).

3.1 Matching demand curves

There is a natural interpretation of our order statistics model in terms of bidders' mean demand curves in the following sense. Fixing a uniform price p, the expected number of units demanded by bidder i is $(1 - F^i(p))m_i$.[30] The specification of the mean demand curve is therefore flexible because F^i is arbitrary, while the distribution of demand curves about the mean is determined by the properties of the order statistic model.[31]

[29] With a reserve price the "only if" is still valid, but the "if" may fail. Nonetheless the competition faced for unit k is against opponents' aggregate demand for $m - k + 1$ units.

[30] For a fixed price, the number of units demanded out of a maximum of m is a random variable with a binomial distribution with probability of "success" given by $1 - F^i(p)$.

[31] For example, the observations in Footnote 30 imply that the variance of the number of units demanded at price p must be $F^i(p)(1 - F^i(p))m_i$, or large for intermediate prices and small for prices near 0 or 1.

Our model can account for stochastic demand. For simplicity we assume that the underlying distributions F^i are continuous, but all our results go through in the presence of a mass point at 0, the lowest possible value.[32] Since per-unit values are given by order statistics of m_i draws from this distribution, this implies that each agent's total demand — the number of units for which she has a strictly positive value — is given by a binomial distribution. This model can fit settings in which demand is roughly known but subject to some small variance around its maximum.

4 Equilibrium of the last accepted bid

We begin by deriving first order conditions for best response behavior, under the assumption that equilibrium bid functions are appropriately well behaved.[33] Using these first order conditions, we obtain closed-form equilibrium bidding strategies in balanced markets when all bidders are symmetric, and when there are two asymmetric bidders.[34]

A bidder's interim utility is the probability-weighted sum of her utility for ex post allocations. Conditional on receiving exactly k units, two events determine the bidder's ex post utility: either her bid for unit k, b^i_k sets the market clearing price, or the market clearing price is below b^i_k and above b^i_{k+1}.[35] Let q_i denote the quantity bidder i receives. Taking as given her opponents' bid functions b^{-i}, bidder i's interim utility is given by

$$\begin{aligned} u^i_k\left(b; v^i\right) = & \sum_{k=1}^{m_i} \left(\sum_{\ell=1}^{k} v^i_k \right) \Pr\left(q_i = k\right) \\ & - k b^i_k \Pr\left(q_i = k, b^i_k \text{ sets price}\right) \\ & - k \mathbb{E}\left[b^{-i}_{m-k+1} \middle| q_i = k, b^{-i}_{m-k+1} \text{ sets price}\right] \Pr\left(q_i = k, b^{-i}_{m-k+1} \text{ sets price}\right). \end{aligned} \tag{2}$$

The relevant events have simple expressions in terms of the distribution of opponents' bids:[36]

- Bidder i wins k units when her bid for her k^{th} unit is larger than her opponents' bid for their $m-k+1^{\text{th}}$ aggregate unit, and her bid for her $k+1^{\text{th}}$ unit is less than her opponents' bid for their $m-k^{\text{th}}$ aggregate unit. Then

$$\Pr\left(q_i = k\right) = H^{-i}_{m-k+1}\left(b^i_k\right) - H^{-i}_{m-k}\left(b^i_{k+1}\right).$$

[32] Indeed, some of our results are strengthened in the presence of such a mass point. Our theoretical results rely on results developed in the study of first price auctions. Equilibrium uniqueness in a first price auction, for example, is guaranteed when the distribution of values is log-supermodular, or when there is a mass point at the minimum possible value [Lebrun, 2006].

[33] We verify this well-behavedness in Section 5.

[34] In Section 5 we show that these equilibria are unique, subject to natural qualifications.

[35] For simplicity of notation, we do not separately model the endpoint conditions of receiving 0 or $m_i + 1$ units. We assume that the bid for the 0^{th} unit, b^i_0 is sufficiently high, and the bid for the $m_i + 1^{\text{th}}$ unit, $b^i_{m_i+1}$, is sufficiently low, generating bid distribution functions so that the bidder receives between 0 and m_i units.

[36] Since we are constraining attention to well-behaved bid functions in this section, it is not mathematically relevant whether our probabilities are given in terms of strict or weak inequality. We show later that it is without loss of generality to ignore mass points in equilibrium bid functions.

- Conditional on winning k units, the bid b_k^i sets the market clearing price whenever it is the m^{th} highest bid. Then b_k^i sets the market clearing price when her opponents' bid for their $m-k^{\text{th}}$ aggregate unit exceeds her bid for her k^{th} unit, $b_{m-k}^{-i} > b_k^i$. Then

$$\Pr\left(q_i = k, b_k^i \text{ sets price}\right) = H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_k^i\right).$$

- Conditional on winning k units, the opponents' bid for the $m-k^{\text{th}}$ aggregate unit sets the market clearing price whenever it is the m^{th} highest bid. From bidder i's perspective, bid b_{m-k}^{-i} sets the market clearing price whenever b_{m-k}^{-i} is between b_k^i and b_{k+1}^i. Then

$$\mathbb{E}\left[b_{m-k+1}^{-i}\middle| q_i = k, b_{m-k+1}^{-i} \text{ sets price}\right]\Pr\left(q_i = k, b_{m-k+1}^{-i} \text{ sets price}\right) = \int_{b_{k+1}^i}^{b_k^i} b dH_{m-k+1}^{-i}(b).$$

Lemma 6 uses these expressions to give interim utility as

$$u_k^i\left(b^i; v\right) = \sum_{k=1}^{m} v_k H_{m-k+1}^{-i}\left(b_k^i\right) - \left(H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_k^i\right)\right) k b_k^i$$
$$- k\int_0^{b_k^i} b dH_{m-k}^{-i} + (k-1)\int_0^{b_k^i} b dH_{m-k+1}^{-i}.$$

This allows for a surprisingly compact representation of the first order conditions for optimality.

Lemma 1 (LAB first order conditions)**.** *In LAB, bidder i's first order condition for her bid for her k^{th} unit is*

$$\underbrace{\left(v_k - b_k\right) dH_{m-k+1}^{-i}\left(b_k\right)}_{\textit{benefits}} = \underbrace{\left(H_{m-k+1}^{-i}\left(b_k\right) - H_{m-k}^{-i}\left(b_k\right)\right) k}_{\textit{costs}}. \tag{3}$$

Lemma 1 cleanly compares the costs and benefits of a marginal increase in bid. The benefits, on the left-hand side of equation (3), are exactly the margin on unit k times the marginal increase in probability of winning unit k. An increase in the bid for unit k cannot affect the probability of winning $k-1$ units or $k+1$ units. The costs, on the right-hand side of equation (3), are exactly the probability that bid b_k^i sets the market clearing price multiplied by k, to account for the fact that the small increase in bid is paid k times over, once for each unit won.

We now use equation (3) to obtain explicit forms for equilibrium bids in two natural cases of market balance: symmetric bidders, and two asymmetric bidders. Market balance allows the generic opponent bid distribution functions H^{-i} to be conveniently factored.

4.1 Symmetric bidders in a balanced market

Recall that a balanced market is one where $m = \sum_{j\neq i} m_j$. This implies that each bidder faces exactly m bids from opponents in equilibrium. The case where two bidders each demand m units, examined in Section 4.2 below, is one such example.

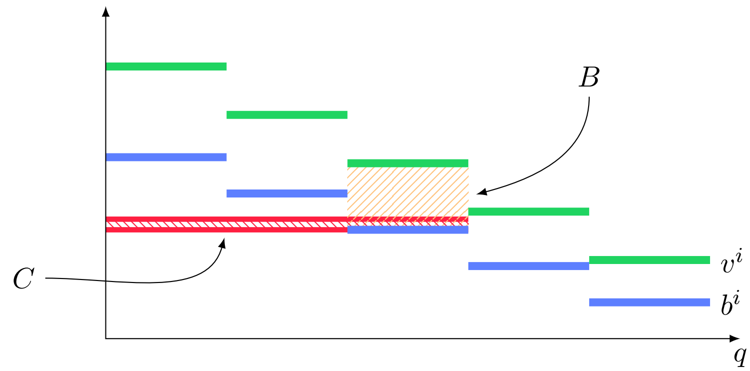

Figure 3: Costs and benefits of a small increase in the bid for unit $k = 3$. Region B is the gains from a small increase in bid, the area between the value and bid for unit k; it is a discrete benefit, obtained with marginally greater probability. Region C is the costs from a small increase in bid, the area between the original and increased bids for unit k; it is a marginally greater cost, incurred with discrete probability. Note that the increase in payment is multiplied by the number of units which are obtained. Except for this multiplier, the tradeoffs are identical to a standard first price auction.

When bidders are symmetric, each bidder's value for her k^{th} unit is distributed according to the k^{th} order statistic from the same distribution F (i.e., $v_k^i \sim F_{(k)}$ for each i). Building on the example in Section 2, suppose that each opposing bidder shades her bid consistently for every unit in the sense that there is some increasing $b(\cdot)$ such that for all agents j and all units k, $b_k^j = b(v_k^j)$. Let $\varphi(b)$ be the inverse of $b(v)$.[37] Symmetry of the underlying value distributions and strategies implies symmetry of equilibrium bid distributions.

When bidder i uses the same bid function for all units and her values are distributed as order statistics, her bids will also be distributed as order statistics. In particular,

$$\Pr\left(b_k^i \leq b\right) = \Pr\left(v_k^i \leq \varphi\left(b\right)\right) = F_{(k)}\left(\varphi\left(b\right)\right).$$

Let $F_{(k:\tau)}$ be the distribution of the k^{th} order statistic taken from τ draws. Independence of bidder values and symmetry of the bid distributions then implies that the distribution of the k^{th} highest bid from a set of bidders with aggregate demand τ is given by $F_{(k:\tau)} \circ \varphi$. Then bidder i's competition for unit k is determined by her opponents' bid for aggregate unit $m - k$, which has distribution $F_{(k:m)} \circ \varphi$.

Equation (3) shows that each bidder's incentives depend only on the difference of two order statistics distributions, which depends on k in a straightforward way: at the optimum, a bid for a particular unit is determined only by the probability that this is exactly the bidder's marginal unit. These are exactly the incentives in a standard single-unit first price auction, leading to Proposition 1.

Proposition 1 (Equilibrium in symmetric LAB)**.** *If marginal values for each bidder are the order*

[37] We show later that this inverse exists.

statistics from independent draws from F the equilibrium bids for bidder i in the LAB auction are

$$\mathbf{b}^i(\mathbf{v}_i) = \left(\frac{1}{F\left(v_k^i\right)} \int_0^{v_k^i} x f\left(x\right)\, dx \right)_{k\in\{1,..,m_i\}}.$$

Proof. Applying facts about order statistics, the costs in equation (3) are given by

$$\left(H_{m-k+1}^{-i}\left(b\right) - H_{m-k}^{-i}\left(b\right)\right) k = \binom{m}{m-k} \left(1 - F\left(\varphi\left(b\right)\right)\right)^{m-k} F\left(\varphi\left(b\right)\right)^k k.$$

The benefits in equation (3) are given by

$$\left(v_k - b\right) dH_{m-k+1}^{-i}\left(b\right) = \left(v_k - b_k\right) \binom{m}{m-k} F\left(\varphi\left(b\right)\right)^{k-1} \left(1 - F\left(\varphi\left(b\right)\right)\right)^{m-k} k f\left(\varphi\left(b\right)\right) d\varphi\left(b\right).$$

Then equation (3) becomes

$$\left(v_k - b_k\right) d\varphi\left(b\right) = F\left(\varphi\left(b\right)\right).$$

It is clear that the candidate equilibrium bid, $b(v_k^i)$, satisfies the k^{th} first order condition. Standard arguments establish that the partial derivative is negative (positive) for $b(v')$ when $v' > v_k^i$ ($v' < v_k^i$), which also means that the objective must be lower at the end points, $b(v_{k-1}^i)$ and $b(v_{k+1}^i)$, given by the monotonicity constraint.

□

The equilibrium in Proposition 1 is efficient, and hence standard arguments imply that the expected payment should be equal to the Vickrey payment. To see this, let $Y_{(k)}$ denote the k^{th} order statistic from m draws from F. Consider the event that $b(v_k^i)$ is the last accepted bid (i.e., $Y_{(m-k)} \geq v_k^i \geq Y_{(m-k+1)}$). In this event the bidder pays $kb(v_k^i)$, which is the expected payment made for k units in a Vickrey auction conditional on this event because

$$\begin{aligned} &\sum_{j=1}^{k} \mathbb{E}\left[Y_{(m-k+j)} \middle| Y_{(m-k)} \geq v_k^i \geq Y_{(m-k+1)} \right] \\ &\quad = \sum_{j=1}^{k} \mathbb{E}\left[Y_{(j:k)} \middle| v_k^i \geq Y_{(1:k)} \right] = k\mathbb{E}\left[Y \middle| v_k^i \geq Y \right] = kb\left(v_k^i\right), \end{aligned} \tag{4}$$

where the notation $Y_{(j:k)}$ denotes the j^{th} highest value out of k independent draws from F. To understand the first equality, observe that conditional on the event $Y_{(m-k)} > v_k^i > Y_{(m-k+1)}$ the first $m-k$ random variables provide no additional information about the last k random variables, so the expectation reduces to one involving just the last k. Recall that in the Vickrey auction a bidder who wins k units in this environment would be required to pay the sum of the k rejected bids made by the opponents. In other words, the bids in this equilibrium are set so that the expected payment equals the expected Vickrey auction payment conditional on the event that the bid determines the payment.

4.2 The asymmetric $2 \times m$ case

When marginal values for two bidders, who each demand all m units, are drawn from different distributions, the equilibrium will in general no longer be efficient or have closed-form expressions, as is the case in asymmetric first price auctions for a single good. However, given that first order conditions in this auction take forms very similar to that of the first price auction for a single good, many of our earlier results carry over.

Most of the literature on asymmetric first price auctions focuses on the two-bidder case. An analogous case in this model is set up as follows. Suppose that there are $n = 2$ bidders, $i \in \{1, 2\}$, who each value marginal units according to m draws each from F^i, where $F^1 \neq F^2$. Suppose that i's bids for each marginal unit are determined by the increasing, differentiable function $b^i(v^i_k)$ with inverse $\varphi^i(b)$, common across all units. Then bidder i wins unit k if and only if bidder $-i$ does not win unit $m - k + 1$. It follows that $H^{-i}_{m-k} = F^{-i}_{(m-k)}$, and equation (3) becomes

$$(v_k - b)\, f^{-i}_{(m-k+1)} \left(\varphi^{-i}(b)\right) d\varphi^{-i}(b) = \left(F^{-i}_{(m-k+1)} \left(\varphi^{-i}(b)\right) - F^{-i}_{(m-k)} \left(\varphi^{-i}(b)\right)\right) k.$$

As in the proof of Proposition 1, this reduces to the expression studied in Maskin and Riley [2000]. If $b^1(v)$ and $b^2(v)$ are the equilibrium bid functions from the first price auction involving two bidders with corresponding value distributions F^1 and F^2, then setting $b^i_k = b^i(v^i_k)$ will satisfy bidder i's first order condition for good k when $\mathbf{b}^i(\mathbf{v}^i) = \left(b^i(v^i_k)\right)_{k\in\{1,\ldots,m\}}$.

Proposition 2 (Equilibrium in asymmetric LAB)**.** *With two bidders whose m marginal values are the order statistics from F^1 and F^2, let $b^1(v)$ and $b^2(v)$ be the equilibrium bid functions in the first price auction for a single unit with two bidders whose values are distributed according to F^1 and F^2. That is,*

$$\mathbf{b}^i\left(\mathbf{v}^i\right) = \left(\frac{1}{F^{-i}\left(v^i_k\right)} \int_0^{v^i_k} x f^{-i}(x)\, dx\right)_{k\in\{1,\ldots,m_i\}}.$$

Then the strategies $\mathbf{b}^i(\mathbf{v}^i) = \left(b^i(v^i_k)\right)_{k\in\{1,..,m\}}$ constitute an equilibrium of the LAB auction.

Proof. Analogous to the proof of Proposition 1. □

Given the relation between equilibrium bidding in the LAB auction and bidding in asymmetric first price auctions, a number of results follow immediately. Instead of exhaustively listing them here, we emphasize their interpretation in this model. Recall that we may interpret the function $(1 - F^i(p))m$ as bidder i's mean demand curve. It follows from the previous section, that $(1 - F^i(\varphi^i(b)))m$ represents the mean number of bids placed by bidder i that exceed b in equilibrium, referred to as the mean quantity demanded in equilibrium. Bidder $-i$'s mean residual supply curve is therefore $F^i(\varphi^i(b))m$, which is proportional to the equilibrium bid distribution of a bidder with type distribution F^i in a first price auction.

The stochastic dominance properties used in the asymmetric first price auction literature have immediate analogues to properties of the mean demand curves in this model. For example, bidder

i having weakly higher mean demand than bidder $-i$ at each price is equivalent to F^i first order stochastically dominating F^{-i}. An implication from the first price auction literature is that bidder i's mean quantity demanded weakly exceeds bidder $-i$'s in equilibrium [Kirkegaard, 2009, Corollary 1]. The stronger distributional ordering property of reverse hazard rate dominance can be stated as follows.

Definition 1 (Reverse hazard rate dominance)**.**

$$F \succeq_{\mathrm{rh}} G \iff \frac{d}{dx}\frac{F(x)}{G(x)} \geq 0,\ \forall x.$$

When F and G admit densities at x, this implies $f(x)/F(x) \geq g(x)/G(x)$. If $F^i(x)m$ is the mean residual supply curve that bidder i would present to bidder $-i$ if she were to bid her value for each unit, then $xf^i(x)/F^i(x)$ is the elasticity of that supply curve. The reverse hazard rate condition can then be interpreted as requiring that these elasticities are ordered. From Proposition 3.5 of Maskin and Riley [2000] we can therefore conclude that this ordering of elasticities is sufficient to order the bid curves of the bidders, meaning $F^i \succeq_{\mathrm{rh}} F^{-i}$ implies $b^i(v) < b^{-i}(v)$ or $\mathbf{b}^i(\mathbf{v}) < \mathbf{b}^{-i}(\mathbf{v})$. This is the well-known "weakness leads to aggression" result.

Finally, we make one more connection to work on investment incentives in single unit auctions. In their Proposition 3 Arozamena and Cantillon [2004] show that if one bidder is given the opportunity to "upgrade" their type distribution ex ante by making it stronger with respect to hazard-rate dominance, the investment incentives are stronger in the second-price auction than in the first-price auction. Furthermore, their Proposition 4 shows that investment incentives are optimal in the second-price auction. Upgrading the distribution has a natural interpretation in our model. It is equivalent to a bidder in our model investing to increase her mean demand curve in such a way as to weakly increase the elasticity of the mean residual supply curve at every point. From the Arozamena and Cantillon [2004] results we get immediate comparisons of the investment incentives in the LAB auction to those in the Vickrey auction, which is the extension of the single-unit second price auction to this environment.

4.3 The general case

If the market is either balanced and bidders have symmetric demands or there are two bidders with asymmetric demand for all units, we can identify equilibrium strategies with a corresponding first price auction. This is no longer true in the general case where the market is either unbalanced or there are more than two asymmetric bidders. A common property of the equilibria in both of the previous sections is that there exists a univariate function which bidder i uses to determine the bids on all of her marginal units from their marginal values. In general this property, which allows for the reduction to a first price auction, does not hold in equilibrium, and bidders may shade their bids on marginal units differently depending on the unit to which the bid is on.

Despite not being able to explicitly characterize equilibrium strategies in the general case, we show in the subsequent section that we can utilize techniques from the first price auctions literature to establish that some key properties still hold. For example, we present an unbalanced, symmetric demand case in which we can prove a uniqueness result for equilibrium strategies using an argument that closely resembles the uniqueness argument typically given for the equilibrium of a first-price auction.

For now, we provide an existence result for the general model.

Proposition 3 (Equilibrium existence in LAB)**.** *With $n \geq 2$ bidders $i \in \{1, \dots, n\}$, where bidder i's m_i marginal values are the order statistics from the distribution F^i, the LAB auction admits a pure strategy Bayesian Nash equilibrium.*

Proof. This follows from Corollary 5.2 in Reny [2011].[38] □

5 Equilibrium properties

The previous section shows that there is a close connection between the equilibrium of the first price auction and that of the LAB auction. In this section we discuss properties of these equilibria, focusing on uniqueness and information transmission. These properties lie at the heart of the practical value of the LAB auction model: separation ensures that, in equilibrium, first order conditions are satisfied with equality, while uniqueness ensures that bidders in practice will behave comparably across auction implementations.

We first establish the existence of separating equilibria, in the sense we define below. Leveraging results from single-unit auctions, we then prove that separating equilibrium is unique. While this does not preclude the existence of non-separating equilibria, we show that non-separation occurs only when the monotonicity constraint binds.[39] Thus, while equilibria in separating strategies may be computed in a straightforward manner by applying results from single-unit auctions, equilibria in non-separating strategies are computationally intractable. From an empirical perspective, uniqueness of separating equilibrium implies that the natural candidate for counterfactual analysis is well-determined.

5.1 Separation

First price auctions do not typically admit equilibria in which participants with different values pool on identical bids. The intuition is straightforward and applies to many other single-unit auction contexts: the presence of a mass point in one bidder's distribution of bids implies a discontinuity in the density of the distribution of the winning bid, generating disincentives to opponents submitting bids either just above or just below the mass point. The lack of one opponent's bids in the neighborhood of the mass point generates a feedback effect by which other opponents are even more disincentivized to bid in this neighborhood, and this behavior cannot be sustained in equilibrium.

[38] Reny [2011] investigates the FRB auction. With regard to existence (although not, as argued above, the structure of equilibrium) the arguments do not change in a substantive way.

[39] Contrariwise, we do not know of any constructions of equilibria in non-separating strategies. We conjecture that no such equilibria exist, but this remains an open question.

Any monotone strategy that does not generate mass points in the bid distribution must be such that all types submit different bids, and therefore equilibrium is separating.

In multi-unit auctions there is the potential for probability-zero pooling behavior, in which multiple types submit the same bid but any given bid is submitted with probability zero. For example, a bidder who demands two units could bid her average value for both units. Then the distribution of her bids is exactly the distribution of underlying types, which is massless, but any given bid is submitted by more than one type of bidder. Fundamentally this effect arises when the type space is of a higher dimension than the (realized) bid space, which is a possibility when bids for distinct units are co-determined, for example by passing through the monotonicity constraint. Such co-determination is unit to multi-unit auctions and cannot occur in single-unit auctions, as bids are uni-dimensional.

Following from properties of single-unit auctions it may at first appear obvious that the LAB auction leads to equilibria which are separating in both of the above senses. However, as we show in Section 6, equilibria in the FRB auction are generically non-separating in the first sense, while equilibria in the PAB auction are generically non-separating in the second sense.

To begin, we define the notion of a strictly separating strategy.

Definition 2 (Strictly separating bids)**.** *A bid function b^i is* strictly separating *if the inverse bid correspondence is at most single-valued; that is, for all type profiles v^i,*

$$\#\left\{v:\ b^i\left(v\right)=b^i\left(v^i\right)\right\}=1.$$

An equilibrium bid profile $(b^i)_{i=1}^n$ is strictly separating if b^i is strictly separating, for all $i \in \{1,\ldots,n\}$.

In equilibria that are strictly separating bid curves preserve all information about the marginal values. In other words, bid curves are invertible given bid data — a useful property for empirical work. Because the monotonicity constraint on bid curves never binds, the optimization problem can be solved bid-by-bid. This reduces the complexity of the bidder's problem as well as the complexity of computationally solving for equilibrium.

Observe that in the case with two bidders and m units the equilibrium described in Section 4 satisfies strict separation, since the event that a bidder submits two marginal bids that are equal to one another has zero measure.

Corollary 1. *Suppose there are n symmetric bidders, or $n = 2$ bidders with potentially asymmetric value distributions, with demands satisfying market balance. There exists an equilibrium of the LAB auction in which bids are strictly separating.*

Corollary 1 follows immediately from the bid functions derived in Propositions 1 and 2. In certain cases where market balance is not satisfied, and for which we do not provide an explicit equilibrium characterization, it is possible to demonstrate separation from first principles. We now look at the $n \times 2$ case, in which n symmetric bidders each demand $m_i = 2$ units, with $m = 2$ units for sale.[40]

We focus on symmetric equilibria. Let $G_1(b)$ and $G_2(b)$ be the equilibrium distributions of *each bidder's* first and second bids. Define $F_{(k)}^{-1}$ as the inverse of the distribution of the k^{th} order statistic from F, and let $\varphi_k = F_{(k)}^{-1} \circ G_k$ be the inverse bid function for unit k.[41] After isolating $g_1 \equiv dG_1/db$ and $g_2 \equiv dG_2/db$, when the bid monotonicity constraint does not bind the first order conditions require that

$$g_1(b) = \frac{1}{n-1}\left(\frac{2G_1(b)}{\varphi_2(b) - b}\right), \text{ and}$$
$$g_2(b) = \frac{G_2(b) - G_1(b)}{\varphi_1(b) - b} - 2\left(\frac{n-2}{n-1}\right)\left(\frac{G_2(b) - G_1(b)}{\varphi_2(b) - b}\right).$$

Since g_2 is a probability density, it must be that $g_2(b) \geq 0$. This requires that

$$\frac{1}{\varphi_1(b) - b} \geq 2\left(\frac{n-2}{n-1}\right)\frac{1}{\varphi_2(b) - b}.$$

Then $\varphi_2(b) \geq \varphi_1(b)$ when $n \geq 3$, and this inequality is strict when $n > 3$.[42] Note that the bid monotonicity constraint implies that $G_2(b) \geq G_1(b)$ for all b.

Proposition 4. *Suppose that there are n symmetric bidders, each with demand for $m_i = 2$ units, and $m = 2$ units are available for sale. Then in all symmetric equilibria of the LAB auction, bids are strictly separating.*

Proof. See Appendix B. □

The inequality at the heart of Proposition 4 exposes a distinction between models with and without market balance: when markets are unbalanced, maximum bids need not be equal across units. This is Corollary 2, which is useful in our examination of equilibrium uniqueness.

Corollary 2. *When $n > 3$ and equilibrium bids are monotone in values, $b_1(1) > b_2(1)$.*

With two bidders, a bid for unit k competes only against an opponent's bid for unit $m - k + 1$; then distributions of bids for these units should have the same upper bound. With more agents this intuition fails. When there are only two units, for example, a bid for the second unit competes only against opponents' first-unit bids, while a bid for the first unit competes against opponents' bids for both the first and second units. Then it is no longer true that the bid distributions must be equal, only that the support of second-unit bids is a subset of the support of first-unit bids.

[40] This setting satisfies market balance if and only if $n = 2$. Although we do not do so here, it can be shown that when $n > 2$, equilibrium bids must depend on the unit for which they are being submitted.

[41] The existence of a well-defined φ_k is formally established in the proof of Proposition 4. φ_k is well-defined as long as b_k is strictly monotone. If b_k is monotone but not strictly so, there are mass points in the distribution of market clearing prices, which implies a profitable deviation for some bidder.

[42] This implication is not bidirectional. Since market balance is satisfied when $n = 2$, we have already established that $\varphi_2(b) \geq \varphi_1(b)$ in this case.

5.2 Uniqueness

Several authors have investigated the uniqueness of equilibrium bidding strategies in the first-price auction, notably Maskin and Riley [2003], Bajari [2001], and Lebrun [2006]. Their arguments for uniqueness are based on analyses of the system of differential equations in the inverse bid functions derived from first order conditions. In our derivation of equilibrium for the LAB auction with symmetric bidders satisfying market balance, we show that under the assumption that the opponent uses the same univariate bid function for each marginal unit we recover the same system of differential equations. It follows that if the bidders are restricted to using the same bid function for each marginal unit the existing uniqueness results for the first-price auction apply in our setting.

In this section, we constrain attention to two models: first, the $2 \times m$ model, where there are $n = 2$ bidders and market balance is satisfied; second, the symmetric $n \times 2$ model, where there are n symmetric bidders, each with demand for $m_i = 2$ units and $m = 2$ units are available. We extend our equilibrium analysis to show uniqueness over a larger set of strategies. We show that the equilibrium identified in the previous section is unique among all separating strategies. More precisely, we show that whenever the corresponding first price auction admits a unique equilibrium, the equilibrium we have identified is unique among separating strategies.

5.2.1 Uniqueness in the balanced $2 \times m$ model

The arguments for uniqueness in the first price auction given in the literature typically use the same intermediate arguments.[43] First, one shows that the largest equilibrium bid (or smallest in the case of procurement) is the same for every bidder. Second, one defines a system of ordinary differential equations involving inverse bid functions. The equations in the system are shown to be necessary and sufficient for optimality and also to satisfy the Lipschitz condition at every bid but the lowest. The initial value problem starting from a particular highest bid therefore has a unique solution due to the fundamental theorem of ordinary differential equations (FTODE). Third, one shows that if $\bar{b}$ and $\widetilde{b}$ are two initial values with $\bar{b} < \widetilde{b}$ then the solutions to the initial value problem using $\bar{b}$ are greater than those to the problem using $\widetilde{b}$ at every interior b. Finally, one shows with an additional assumption about the problem at the lowest bid that the second and third results imply that there can only be one highest bid yielding a solution that is also an equilibrium.

To establish uniqueness of the LAB equilibrium among separating strategies in our model, we follow the first two steps above but then appeal to the uniqueness of the corresponding first price auction solution to complete the proof. We restrict attention to separating strategies, because our argument relies on the analysis of a system of differential equations that is only valid for separating strategies. Allowing the monotonicity constraint to bind for arbitrary bids leads to a system of equations that is substantially more difficult to analyze and, as we argue, less likely to describe

[43] We refer to Lebrun [2006] for a discussion of uniqueness results in the first price auction literature and the assumptions required to prove uniqueness. There is a unique equilibrium in the asymmetric first price auction under fairly general conditions, but as argued in Lebrun [2006] some prior proofs have relied on unjustified uses of l'Hôpital's rule.

observed bidding behavior.

The most important step in our argument is to establish that for any unit and any bidder, the highest bid submitted for this unit is equal to the upper bound of the support of all bids — the maximum bid submitted is independent of unit and bidder. This does not follow directly from the analogous argument in the first price auction, although there are similarities. The added difficulty here arises from the facts that there is a monotonicity constraint on bids, and that the probability that a bid on unit k is utility-relevant depends on the distribution of two of the opponent's bids. After establishing an intermediate lemma, we prove this equal upper bid result in Lemma 3.

Lemma 2 (Constrained bids strictly between optimal unconstrained bids)**.** *Suppose that bidder i with type v^i submits a constant bid $b^i_{\{k,\dots,k+a\}}$ for units $k,\dots,k+a$ and let $b^i_l(v^i_l)$ and $b^i_s(v^i_s)$ with $l,s\in\{k,\dots,k+a\}$ be respectively any of the bidder's largest and smallest unconstrained bids for these units. Then $b^i_l(v^i_l)>b^i_s(v^i_s)$ implies $b^i_l(v^i_l)>b^i_{\{k,\dots,k+a\}}>b^i_s(v^i_s)$.*

Proof. See Appendix B. □

Lemma 2 gives the natural result that if the a bidder's monotonicity constraint is binding, so that her ideal (unconstrained) bid differs from the bid she submits, her constrained bid is strictly between the maximum and minimum optimal unconstrained bids.

Lemma 3 (Equal upper bids)**.** *In equilibrium, there is a $\overline{b}$ such that for all i and k, $\overline{b}^i_k=\overline{b}$.*

Proof. First, it cannot be that $\overline{b}^i_k=\overline{b}^i$ for all i and k but $\overline{b}^i\neq\overline{b}^{-i}$ because the type of bidder who submits the higher maximum bid could lower all of her bids and reduce her payment without reducing the probability of winning any units. Therefore if the lemma is false $\overline{b}^i_k>\overline{b}^i_{k+1}$ for some k and i. Let $\hat{k}$ to be the lowest k for which $\overline{b}^i_k>\overline{b}^i_{k+1}$.

We claim that $\overline{b}^{-i}_{m-\hat{k}+1}=\overline{b}^{-i}_{m-\hat{k}}=\overline{b}^i_{\hat{k}}$. It must be that $\overline{b}^{-i}_{m-\ell+1}\leq\overline{b}^i_\ell(=\overline{b}^i_{\hat{k}})$ for all $\ell\leq\hat{k}$ because otherwise the type of bidder $-i$ placing these bids could weakly reduce all of these bids without reducing the probability of winning any of the items. If $\overline{b}^{-i}_{m-\ell+1}\leq\overline{b}^{-i}_{m-\hat{k}+1}<\overline{b}^i_{\hat{k}}$ for all $\ell\leq\hat{k}$, then bidder i should respond by reducing his maximum bids on the first $\hat{k}$ units for the same reason. Hence, $\overline{b}^{-i}_{m-\hat{k}+1}=\overline{b}^i_{\hat{k}}$. Now $\overline{b}^{-i}_{m-\hat{k}+1}=\overline{b}^{-i}_{m-\hat{k}}$ follows because otherwise $\overline{b}^{-i}_{m-\hat{k}}>\max\{\overline{b}^{-i}_{m-\hat{k}+1},\overline{b}^i_{\hat{k}+1}\}$ and $\overline{b}^{-i}_{m-\hat{k}}$ can be reduced without lowering the probability of winning or violating monotonicity.

Finally, for $v_{m-\hat{k}}$ close to 1, $b^{-i}_{m-\hat{k}}(v_{m-\hat{k}})\leq\overline{b}^i_{\hat{k}+1}<\overline{b}^i_{\hat{k}}=\overline{b}^{-i}_{m-\hat{k}+1}$. Lemma 2 then implies that the optimal choice of constrained bid for bidder 2 is strictly below $\overline{b}^i_{\hat{k}}$, which is a contradiction. □

Having established that there is a common maximum bid for all units and bidders, we next describe the system of differential equations we evaluate. As with first price auctions, the arguments are made simpler by writing the differential equations in terms of an unknown derivative with respect to a bid distribution (see, e.g., Lebrun [2006]). Recall that in this section we assume that bidders use separating strategies. This implies that $\varphi^i_k(b)\leq\varphi^i_{k+1}(b)$ for all k and i. Consequently, the distribution of the k^{th} bid of bidder i is $F^i_{(k)}(\varphi^i_k(b))$. Furthermore, bidder $-i$'s first order condition with respect to her $m-k+1^{\text{th}}$ bid becomes

$$\left[\varphi_k^i\right]'(b)\, f_{(k)}\left(\varphi_k^i(b)\right)\left(v_{m-k+1}^{-i}-b\right)-(m-k+1)\left(F_{(k)}^i\left(\varphi_k^i(b)\right)-F_{(k-1)}^i\left(\varphi_{k-1}^i(b)\right)\right)=0.$$

We create a system of $2m$ differential equations out of the first order conditions for each bid by each bidder. Instead of writing the system in terms of unknown inverse bid functions, we write it in terms of unknown bid distributions as follows.

Definition 3 (LAB differential system)**.** *Let $H_k^i \equiv F_{(k)}^i \circ \varphi_k^i$, $H_0^i(b) \equiv 0$, $\varphi_{m-k+1}^{-i} \equiv F_{(m-k+1)}^{-i,-1} \circ H_{m-k+1}^{-i}$, and $\bar{b} \in (0,1)$ be given. For $k \in \{1, \ldots, m\}$, $i \in \{1, 2\}$, and $b \in (0, \bar{b}]$, the differential system representing best response behavior is given by*

$$\begin{aligned} \frac{d}{db} H_k^i(b) &= (m-k+1)\,\frac{H_k^i(b)-H_{k-1}^i(b)}{\varphi_{m-k+1}^{-i}-b}, \\ H_k^i\left(\bar{b}\right) &= 1. \end{aligned} \tag{5}$$

This initial value problem involves a system of $2m$ equations — m equations for each of 2 bidders — in $2m$ unknown functions, H_k^i. The following lemma establishes that an equilibrium of the LAB auction is necessarily a solution to this initial value problem.

Lemma 4. *Any equilibrium bid profile in separating strategies must satisfy equation* (5)*.*

Proof. See Appendix B. □

The derivation Lemma 4 is similar to the derivation of well-behavedness properties in first price auctions. The differential system in Definition 3 is inapplicable only if opponent bids are nondifferentiable, which occurs only when bids have either kinks or constant intervals. Constant intervals are ruled out by the presumption of separation, so the the differential system is necessary and sufficient for best response behavior as long as bid functions do not have kinks. Kinks in the bidder i's bid will imply either mass points in bidder $-i$'s bid, which is disallowed, or gaps in the support of bidder $-i$'s bid. If bidder i has a kink in her bid function at b while bidder $-i$ never bids near b, bidder i has a profitable deviation.

Some care must be taken in our argument to deal with the possibility that profitable deviations are not available due to the monotonicity constraint binding. However, because payoffs are continuous in value, if the bid monotonicity constraint is binding for a bidder with value profile v, there is a nearby bidder with value profile $v' \neq v$ facing roughly the same incentives who submits the same monotonicity-constrained bid. This violates separation, and can therefore be excluded from our analysis.

Compare the problem in Definition 3 to the following corresponding one for the first price auction.

Definition 4 (FPA differential system). *Let $H^i \equiv F^i \circ \varphi^i$ and $\bar{b} \in (0,1)$ be given. For $i \in \{1,2\}$ the differential system representing best response behavior is given by*

$$\frac{d}{db} H^i(b) = \frac{H^i(b)}{\varphi^{-i} - b} \qquad (6)$$
$$H^i(\bar{b}) = 1.$$

For an arbitrary $\bar{b}$, because (6) satisfies the Lipschitz condition for all $b \in (0, \bar{b}]$, the FTODE implies there is a unique solution to the initial value problem in Definition 4. Furthermore, when there is a unique equilibrium in the first price auction, a single such $\bar{b}$ yields a solution that also satisfies the boundary condition $H^i(\underline{b}) = F^i(\varphi^i(\underline{b})) = 0$, where $\underline{b}$ is the lowest equilibrium bid.

Since the system in (5) also satisfies the Lipschitz condition for all $b \in (0, \bar{b}]$, the FTODE implies that there is a unique solution to the problem in Definition 3, given $\bar{b}$. But these two solutions must coincide: when (φ^1, φ^2) is a solution to the first price auction problem, the unique solution to the LAB initial value problem can be found by setting $\varphi^i_k = \varphi^i$ for all k and i. The final step is to observe that while Proposition 2 implies that equilibrium value of $\bar{b}$ generates equilibrium solutions to both problems, a different $\bar{b}$ would generate a solution that is not an equilibrium of the first price auction (by uniqueness) and cannot be an equilibrium of the LAB auction.

Proposition 5. *Consider the LAB auction with $n = 2$ bidders, each with demand for $m_i = m$ units. Suppose that bidder i's values are given by m_i ordered draws from the distribution F^i. If there is a unique equilibrium in the first price auction with $n = 2$ bidders drawing values from F^1, F^2, then there is a unique equilibrium of the LAB auction in which the bidders use separating strategies.*

5.2.2 Uniqueness in the symmetric $n \times 2$ model

In the prior section we showed that symmetric equilibria in this case must involve separating strategies. We now evaluate the uniqueness of symmetric equilibria in this context. Since the case with $n = 2$ bidders is covered by the previous analysis we assume that $n \geq 3$. Let $b_1(v_1)$ and $b_2(v_2)$ represent a candidate equilibrium, where $b_1(v) \geq b_2(v)$ for all $v \in (0,1)$. Denote the inverse bid functions by φ_1 and φ_2 respectively.

The argument closely resembles the uniqueness argument for the $n = 2$ bidder, balanced market case given in Section 5.2.1, which in turn resembles uniqueness arguments given for asymmetric first price auctions. A key difference that arises is that with more than two bidders it is no longer necessarily true that there is a common high bid for each marginal unit. One important implication of proving that there *is* a common high bid for each marginal unit is that the initial conditions for the system of differential equations derived from the first order conditions have a single degree of freedom.

We first show that the initial conditions in the $n \times 2$ case have a single degree of freedom as well, despite the fact that $b_1(1) > b_2(1)$ (see Corollary 2). When $b_1(1) > b_2(1)$, there are two distinct intervals of bids between which the first order conditions for the optimal bids change. Bids $b \in [0, b_2(1)]$ compete against first- and second-unit bids made by opponents. That values are given by ordered draws implies $F_{(1)}(v) \equiv F(v)^2$, $H_1(b) \equiv F_{(1)}(\varphi_1(b))$, $F_{(2)}(v) \equiv 2F(v) - F(v)^2$, and $H_2(b) \equiv F_{(2)}(\varphi_2(b))$. The first order conditions for the first- and second-unit bids in this range simplify to

$$\left(\frac{h_2(b)}{H_2(b) - H_1(b)} + \frac{(n-2)\,h_1(b)}{H_1(b)}\right)(v_1 - b) = 1, \text{ and} \tag{7}$$

$$\frac{(n-1)\,h_1(b)}{2H_1(b)}(v_2 - b) = 1. \tag{8}$$

For bids $b \in (b_2(1), b_1(1)]$, the opposing bids are solely for the first unit, changing the win probabilities for a bidder's first unit. The corresponding first order condition for the first unit is

$$\frac{(n-2)\,h_1(b)}{H_1(b)}(v_1 - b) = 1. \tag{9}$$

Since only first-unit bids are submitted in $(b_2(1), b_1(1)]$ and assuming that bidders use symmetric strategies, the bid function that solves (9) can be represented explicitly up to an unknown bid, because it reduces to the best response ODE of a symmetric first price auction with $n-1$ total bidders.[44] If $b_1(1)$ is known, the unique solution to this ODE can be represented as

$$b_1(1) - b_1(v_1)F(v_1)^{n-2} = \int_{v_1}^{1} x\, dF(x)^{n-2}. \tag{10}$$

Equation (10) is an explicit characterization of first-unit bids on the interval $(b_2(1), b_1(1)]$ for a known value of $b_1(1)$. We next argue that the value of $b_2(1)$ is pinned down in equilibrium by $b_1(1)$. Note that the first order condition for the first unit, equation (9), is the same as the one in equation (7) with $h_2(b) = 0$. We also observe that it must be that $h_2(b_2(1)) = 0$ because the density of the second order statistic vanishes at the upper bound of its support. The inverse bid function associated with the solution in equation (10) therefore satisfies equation (7) in a neighborhood of $b_2(1)$. This implies that a necessary condition for the selection of $b_2(1)$ is that it be optimally chosen according to equation (8) where $H_1(b)$ is the bid distribution determined by the initial choice of $b_1(1)$ and equation (10). In other words, at $b_2(1)$ the values of h_1 and H_1 are known up to $b_1(1)$.

In the uniqueness argument given in Section 5.2.1, the second step is to show that given two distinct initial conditions for the ODE derived from two distinct choices for the common high bid, the corresponding solutions to the ODE are monotonic in these initial conditions at all points in the interior of the domain. The same property holds in the $n \times 2$ case, which we record as Lemma 5. Similar to our earlier analysis, we view equations (7) and (8) as an ODE in unknown H_1 and H_2 with domain $(0, b_2(1)]$ and initial conditions determined by the value taken by $b_2(1)$ and the value of $\bar{v}_1 \equiv \varphi_1(b_2(1))$, where φ_1 is determined at $b_2(1)$ by equation (10). Using equations (7) and (8)

[44] See our Definition 4.

and letting $\varphi_k \equiv F_k^{-1} \circ H_k$, this ODE can be expressed as

$$\frac{d}{db} H_2(b) = \frac{H_2(b) - H_1(b)}{\varphi_1(b) - b} - 2\left(\frac{n-2}{n-1}\right)\left(\frac{H_2(b) - H_1(b)}{\varphi_2(b) - b}\right), \text{ and} \tag{11}$$

$$\frac{d}{db} H_1(b) = \frac{1}{n-1}\left(\frac{2H_1(b)}{\varphi_2(b) - b}\right). \tag{12}$$

Equations (11) and 12 can be used to show that distinct solutions to the differential system never meet, except potentially at $v = 0$.

Lemma 5. *Let $\hat{b}_1(1) < \tilde{b}_1(1)$ be two initial choices for $b_1(1)$ and $\hat{b}_2(1) < \tilde{b}_2(1)$ be the corresponding choices for $b_2(1)$. Let $\hat{H}_1$ and $\hat{H}_2$ solve equations* (11) *and* (12) *when the initial condition is $\hat{H}_2(\hat{b}_2(1)) = 1$ and $\hat{H}_1(\hat{b}_2(1)) = F_{(1)}(\hat{v}_1)$, and let $\tilde{H}_1$ and $\tilde{H}_2$ solve equations* (11) *and* (12) *when the initial condition is $\tilde{H}_2(\tilde{b}_2(1)) = 1$ and $\tilde{H}_1(\tilde{b}_2(1)) = F_{(1)}(\tilde{v}_1)$. For all $b \in (0, \hat{b}_2(1))$, $\hat{H}_1(b) > \tilde{H}_1(b)$ and $\hat{H}_2(b) > \tilde{H}_2(b)$.*

Proof. Since the equilibrium bid functions are increasing we have $\hat{H}_k(\hat{b}_2(1)) = 1 > \tilde{H}_2(\hat{b}_2(1))$ for $k = 1, 2$. We show next that this inequality holds for all bids $b \in (0, \hat{b}_2(1)]$. To do this, we rule out that $\hat{H}_k$ crosses $\tilde{H}_k$ for either k at any point in the domain. Let $\hat{b} < b_2(1)$ represent the largest bid at which either $\hat{H}_1$ crosses $\tilde{H}_1$ or $\hat{H}_2$ crosses $\tilde{H}_2$. Consider first the case where $\hat{H}_1(\hat{b}) = \tilde{H}_1(\tilde{b})$ and $\hat{H}_2(\hat{b}) > \tilde{H}_2(\hat{b})$. Using equation (12), this implies that $\hat{h}_1(\hat{b}) < \tilde{h}_1(\hat{b})$, but this implies that $\tilde{H}_1$ crosses $\hat{H}_1$ from above, a contradiction. Similarly, one can show using equation (11) that $\hat{H}_1(\hat{b}) > \tilde{H}_1(\hat{b})$ and $\hat{H}_2(\hat{b}) = \tilde{H}_2(\hat{b})$ imply $\hat{h}_2(\hat{b}) < \tilde{h}_2(\hat{b})$. If it were true that $\hat{H}_1(\hat{b}) = \tilde{H}_1(\hat{b})$ and $\hat{H}_2(\hat{b}) = \tilde{H}_2(\hat{b})$ (i.e., they both crossed together), then we would have to conclude by the FTODE that $\hat{b}_2(1) = \tilde{b}_2(1)$, because the FTODE implies that there is a unique solution to the system in equations (11) and (12) beginning from an initial value at a $\hat{b} \in (0, \hat{b}_2(1)]$. □

Lemma 5 implies that any two solutions to equations (11) and (12) are ordered pointwise in the interior of the domain according to the ordering of the high bids on the first unit. The final step in the uniqueness proof is to use this fact to rule out that there can be more than one valid choice of $b_1(1)$.

In the literature on uniqueness in first price auctions, an additional assumption is required to complete this final step. This may be an assumption that l'Hôpital's rule can be applied to the ODE at the low bid;[45] an assumption that there is a binding reserve price or an atom at the lower end of the support of values [Lebrun, 1999, Maskin and Riley, 2003]; or an assumption about the properties of the value distribution in an interval including the lower bound of the support [Lebrun, 2006]. Each of these approaches apply here as well, using the implications of Lemma 5 and the equation in (12).

Equation (12) implies that for two bids, $b < b'$,

$$\frac{H_1(b')}{H_1(b)} = \exp\left\{\frac{2}{n-1}\int_b^{b'} \frac{dx}{\varphi_2(x) - x}\right\}. \tag{13}$$

[45] Lebrun [2006] points out that this assumption is implicit in Bajari [2001].

As in Lemma 5, let $\hat{b}_2(1) < \tilde{b}_2(1)$ so that $\hat{H}_1(b) > \tilde{H}_1(b)$ and $\hat{H}_2(b) > \tilde{H}_2(b)$ for all $b \in (0, \hat{b}_2(1))$. From equations (13) and Lemma 5, it follows that

$$1 < \frac{\hat{H}_1(b')}{\tilde{H}_1(b')} < \frac{\hat{H}_1(b)}{\tilde{H}_1(b)}. \tag{14}$$

With an atom at the bottom of the distribution F equal to c it must be in equilibrium that $H_1(0) = c^2$, implying that if $\hat{H}_1$ and $\tilde{H}_1$ both derive from equilibrium strategies $\hat{H}_1(0)/\tilde{H}_1(0) = 1$. But this requirement conflicts with equation (14), which bounds this ratio away from one for all $b < b'$. We conclude that $\hat{H}_1$ and $\tilde{H}_1$ cannot both derive from equilibrium strategies.

Proposition 6. *In the LAB auction with n symmetric bidders each with demand for the two available goods determined by two independent draws from the distribution F where $F(0) > 0$, there is a unique symmetric equilibrium with differentiable bid functions satisfying equations* (11) *and* (12)*.*

The restriction in Proposition 6 to choices of F with an atom at the lower endpoint can be replaced with other another assumption such as an assumption on the validity of using l'Hôpital's rule at the lower endpoint as in Bajari [2001]. However, the discussion in Lebrun [2006] suggests that some additional assumption about the bidding behavior at the lower endpoint is needed to prove uniqueness in the first price auction. Given the close relation of our model to the first price uniqueness problem, we do not believe that we can prove uniqueness without such additional assumptions.

Placing an atom at the bottom of the distribution F is natural in many contexts.[46] For example the introduction of a reserve price r, however small, is equivalent to assuming a mass point at r; since our model is translation-invariant, this is essentially equivalent to a mass point at 0. Proposition 6 also guarantees uniqueness when aggregate demand is stochastic, if bidders potentially have zero value for items. Both cases maintain the close connection to the corresponding first price auction, in the well-studied cases of reserve prices or exogenous non-participation.

6 Properties of other auction equilibria

In this section, we contrast the properties of the LAB auction discussed in the previous sections with those of two other common multi-unit pricing rules, the PAB auction and the FRB uniform price auction. As in Section 5, we focus on issues surrounding separation and uniqueness of equilibrium.

In multi-unit auctions with risk-neutral bidders, a bidder's utility for any allocation can be expressed as a sum of distinct per-unit utilities. Utility is modular [McAdams, 2003]: a bidder's utility for allocation q_i is exactly her utility for allocation $q_i - 1$, plus her margin on unit q_i. This implies that bids for different units are co-determined only when the monotonicity constraint is

[46] Our analysis in this paper assumes that F admits a continuous density everywhere on its support, however our results do not substantively change with the introduction of a mass point at the lower bound of the support.

binding.[47] Then when the monotonicity constraint is *not* binding, it is without loss of generality to analyze behavior in these auctions on a per-dimension basis, as a set of m independent optimization problems. Lemma 6 formalizes this separability in the auction formats we examine.

Lemma 6 (Separability of multi-unit auctions). *For each of the FRB, LAB, and PAB auctions there is a profile of utility functions* $((u_k^i)_{k=1}^{m_i})_{i=1}^n$ *such that interim utility can be written as*

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m_i} u_k^i\left(b_k^i, b^{-i}; v_k\right).$$

In each of these auctions, each dimensional utility function u_k^i *satisfies increasing differences in* (b_k, v_k).

Proof. See Appendix C. □

Recall that bids are strictly separating when the inverse bid function, mapping observed bids to underlying values, is well defined. It is immediate that in a strictly separating equilibrium, bids are strictly increasing in value. It is also true that in a strictly separating equilibrium the monotonicity constraint cannot bind. Suppose that the monotonicity constraint binds, so that bidder i's ideal (unconstrained) bid for unit k, b_k^i, is strictly less than her ideal (unconstrained) bid for unit $k+1$, $b_{k+1}^i > b_k^i$. Following Lemma 2, the optimal constrained bid $\hat{b} \in (b_k^i, b_{k+1}^i)$. Since utility is continuous in value, a small change in her underlying type — for example, a small decrease in v_k^i offset by a small increase in v_{k+1}^i — will leave the monotonicity constraint binding, and hold the optimal constrained bid constant. This implies noninvertibility of equilibrium bid functions, and thus in any strictly separating equilibrium the monotonicity constraint cannot (strictly) bind. This is discussed in more detail in Section 6.2.

By way of contrasting behavior in the FRB and PAB auctions with the strict separation we have shown to be possible in the LAB auction, we introduce the notion of partial pooling.

Definition 5 (Partial pooling). *A bid function* b^i *exhibits* partial pooling *if the inverse bid correspondence is multi-valued with positive probability; that is,*

$$\Pr\left(v \in \left\{v' : \#\varphi^i\left(b^i\left(v'\right)\right) > 1\right\}\right) > 0.$$

There is a wedge between strict separation and partial pooling: inverse bids might be multi-valued with zero probability. We are concerned with issues of information confounding, and in particular in situations in which information is obfuscated in equilibrium. If equilibrium bids are non-separating with probability zero, equilibrium is essentially separating, and this distinction is not meaningful.

Our definition of partial pooling is structured to capture two separate pooling effects. In the FRB auction, truthful bidding for the first unit is a weakly dominant strategy. However, we show

[47] These models exhibit the standard IPV mechanism design monotonicity-in-value. Because this is a result and not a constraint, when we refer to binding monotonicity constraints we are referring to monotonicity in quantity.

that there is a range of last-unit valuations such that a bid of zero strictly dominates all others; this occurs because residual competition comes from opponents' small quantities, for which bid distributions are relatively strong. Increasing the bid for the final unit has little marginal effect on the probability of winning the unit but a comparatively strong marginal effect on the expected cost paid for all $m_i - 1$ units, conditional on their being won. In an equilibrium with truthful bids for the first unit, the probability of witnessing any particular bid is zero even though the probability of witnessing a zero bid for an agent's final unit is strictly positive; partial pooling captures this positive-probability noninvertibility.

Partial pooling also captures the information confounding we observe in the PAB auction. In PAB, the bidder is facing increasingly aggressive competition as she considers her bid for higher units: her bid for higher units is against her opponents' bids for lower units. We show that there is generally an incentive for the idealized bid for the first unit to be below the idealized bid for the second unit, violating the bid monotonicity constraint. This implies that, for certain value profiles, bids will be flat for small quantities. Continuity of utility in value implies that this same flat will be realized for nearby value profiles — if, for example, the value for the first unit falls while the value for the second unit rises — and thus upon witnessing a particular flat bid the bidder's value profile cannot be perfectly inverted. Again, this happens in spite of no particular bid profile being submitted with strictly positive probability.

Aside from implications for tractability, information revelation is directly related to efficiency. An efficient mechanism must allocate units to the agents with the highest values. When information is confounded, this is not possible: efficiency entails the mechanism designer knowing which agents have the highest values for the m available units, and standard identification arguments imply that if this is possible, bids must be separating. We thus contrast the FRB and PAB auctions, in which all equilibria exhibit partial pooling and are thus inefficient, with the LAB auction, which we have shown to admit a separable and efficient equilibrium without pooling.

Remark 1. *Any pure strategy equilibrium can be transformed into a monotone pure strategy equilibrium without affecting agents' incentives or payoffs. We therefore restrict attention to equilibria in monotone pure strategies.*

Remark 1 is familiar from other auction contexts. Because bidders with higher values are at least as willing to submit marginally higher bids as bidders with lower values, if bids are nonmonotone in value the bidder with the higher value must be at least indifferent between the pointwise maximum of the two bids and the bid she is submitting. Then bidders with these two value profiles can "swap" their bids for the pointwise maximum and minimum, and their utilities will be (at worst) unaffected.

Under separation and monotonicity, the intuitive notion that bids are optimized unit-by-unit is quickly formalized.

Lemma 7 (Separable bids in separating equilibrium)**.** *In a monotone strictly separating equilibrium* *of the LAB, FRB, or PAB auction models, bidder i's equilibrium bid function can be written as*

$$b^i(v) = \left(b^i_1(v_1), \ldots, b^i_{m_i}(v_{m_i})\right).$$

Taken together, the above results imply that either equilibrium bids can be anlyzed independently, unit-by-unit, or equilibrium exhibits partial pooling. Following the definition of partial pooling, this implies that when bids cannot be analyzed independently equilibrium outcomes must be inefficient, and information is not fully revealed. When comparing the PAB and FRB auctions to the LAB auction, these results allow us to analyze the revelation question dimension-by-dimension and, from these dimensional analyses, to build contradictions which expose the prevalence of partial pooling.

6.1 Partial pooling in the FRB auction

In the FRB auction bids for large quantities are disproportionately unprofitable. A small increase in bid for a large quantity implies that, when this bid is supra-marginal, this increase is paid for each unit won. Because a bidder competes for large quantities against her opponents' small quantities, not only is there an outsized cost associated with increasing this bid but there is also only a relatively small increase in the probability of winning. These incentives balance in favor of a mass point at a bid of zero.

To eliminate pathological cases, we define the notion of a well-behaved equilibrium.

Definition 6 (Well-behaved bids)**.** *A bid function b^i_k is* well-behaved *if $d^t_+ b^i_k/dv^t$ is bounded on $(0,1)$, for all finite t. The bid profile $((b^i)^{m_i}_{k=1})^n_{i=1}$ is well-behaved if b^i_k is well-behaved for all agents i and all units k.*

Lemma 8 (Partial pooling in FRB)**.** *If aggregate supply is $m \geq 2$ and each bidder demands $m_i \geq 2$ units, all well-behaved equilibria of the FRB auction exhibit partial pooling.*

Proof. See Appendix C. □

Lemma 8 is explicitly observed in the two-unit example of Section 2.2. The general principles at work in the order statistic model are readily observed when $m < \sum_{i=1}^n m_i$.[48] Increasing bidder i's bid for unit m_i has two effects: first, it increases the probability that she wins unit m_i. Second, it increases her expected payment conditional on winning $m_i - 1$ units, because occasionally when she wins $m_i - 1$ units her bid on her m_i^{th} unit sets the market clearing price. In the order statistic model, the event that she wins $m_i - 1$ units has probability an order (in the probability distribution) greater than the event that she wins m_i units, so there is an outsized cost associated with increasing her bid.[49] It follows that when her value for unit m_i is relatively low, she will not submit a strictly positive bid for this unit.

[48] When $m \geq \sum_{i=1}^n m_i$ pooling arises due to market degeneracy: there is weak excess supply in the market, and there is no incentive to submit any positive bid.

[49] The sharpness of this intuition follows from the order statistic model. However, pooling on zero bids in the FRB auction can be observed in other informational contexts; see, e.g., Ausubel et al. [2014].

It is worth clarifying the role of well-behavedness in Lemma 8. If the necessary limit (in the proof) does not exist, the Lemma is automatically satisfied: the limit will fail to exist only when the ratio can be discretely positive for b arbitrarily close to 0. This alone is sufficient to indicate that pooling at 0 is advantageous. Thus well-behavedness supports bid density, and provides that $d^{(t+1)}H_{m-k+1}^{-i}(0)$ is finite at the smallest t for which it is nonzero. We do not know of an economic interpretation of this derivative being infinite while all lower derivatives are zero, but nor can we rule it out out of hand.

Remark 2. *With $n = 2$ bidders and $m \geq 2$ units, all equilibria of the FRB auction in weakly dominant strategies exhibit partial pooling. Because truthful reporting for the first unit is weakly dominant, $d^{(t)}H_1^{-i}(0)$ is finite for the lowest t at which it is nonzero, implying that Lemma 8 can be applied directly.*

Remark 2 makes use of a further wrinkle in well-behavedness. It is not necessary that all H_{m-k+1}^{-i} be well-behaved, only that there exists an agent i and a unit k such that H_{m-k+1}^{-i} is well-behaved. This allows for the following proposition.

Proposition 7 (Inefficient equilibrium in FRB)**.** *All equilibria of the FRB auction satisfy one (or more) of the following two properties:*

i. Equilibrium is inefficient.

ii. For all agents i and all units k, $d^{(t)}H_{m-k+1}^{-i}(0)$ is infinite at the lowest t at which it is nonzero.

As discussed above, efficient equilibria must be separating. Then in light of Lemma 8 equilibrium cannot be well-behaved and separating simultaneously. In a natural sense, equilibria that would be expected to be observed must be inefficient.

Point ii. of Proposition 7 appears to be a technicality. We do not know of any equilibrium constructions satisfying point ii. Importantly, equilibrium is inefficient unless all bidders employ the same bidding function for all units. If the underlying value distributions are C^∞ on $[0, 1]$, for point i. to be unsatisfied while point ii. is satisfied there must be $\tau > 0$ such that $d^{(t)}b(0) = 0$ for all $t < \tau$, and $d^{(\tau)}b(0) = \infty$. As mentioned above we cannot rule this out out of hand, but this places a tight restriction on strategic behavior for the FRB auction to be efficient.

6.2 Partial pooling in the PAB auction

As the bid-for quantity increases, the marginal distribution of opponent values shifts upward; the relative lack of competition for small quantities implies partial pooling in the PAB auction. In the case of two bidders a bidder will win unit 1 if and only if her opponent does not win unit m; similarly the bidder will win unit 2 if and only if her opponent does not win unit $m - 1$. Since her opponent's marginal distribution of values for unit $m - 1$ dominates the distribution of values for unit m, the bidder faces less competition for unit 1 than she does for unit 2. Ideally, given a value $v_1 = v, v_2 = v$ she would bid less for unit 1 than for unit 2, violating the monotonicity constraint.[50]

[50] This was explored in a divisible-good context by Woodward [2016].

When the monotonicity constraint is binding, bids cannot be written as profiles of independent bids across the units the bidder demands. Intuition suggests that partial pooling is present: if the bid $b_1 = b, b_2 = b$ is observed, it cannot be known whether this bid has resulted from individual unconstrained bids, or bids that pass through the monotonicity constraint. Then values cannot be inverted out from observed bids, and there is some degree of pooling present in equilibrium.

We now turn attention to *continuous-bid* equilibria, in which bids are continuous functions of value. Continuity is the barest form of equilibrium tractability. In the two-bidder case standard aguments suffice to rule out discontinuities in equilibrium bid functions, but in general this is less clear.[51] Nonetheless, we are able to show that equilibria in continuous bids exhibit partial pooling. Since partial pooling implies inefficiency, as do discontinuous bids, it follows that all equilibria in the PAB auction are inefficient. Further, inasmuch as partial pooling makes equilibrium computation more difficult, and discontinuous equilibria also present computational challenges, these results can be taken as suggesting a general intractability of PAB auction equilibria.[52]

To begin our analysis, we define the notion of a maximal bidder. Lemma 9 implies that at least two maximal bidders exist.

Definition 7 (Maximal bidder)**.** *Bidder i is a* maximal bidder *if for all k, $\bar{b}^i_k = \bar{b}$.*

Lemma 9 (Equal upper bounds)**.** *Let $\bar{b}$ be the maximum bid submitted for any unit, and let $\bar{b}^i_k$ be bidder i's maximum bid for unit k. There are two bidders, i and j, such that $\bar{b}^i_k = \bar{b}^j_{k'}$ for all units k and k'.*

Proof. See Appendix C. □

Maximal bidders feature prominently in our analysis of partial pooling because they are always in competition with one another. The existence of at least two maximal bidders ensures that when bids are relatively high, the model's first order conditions must be satisfied with equality. As we show in Lemma 10, this implies a mass point at the upper bound of the distribution of bids, for at least one agent-unit tuple.

Lemma 10 (Monotonicity of distributional differences)**.** *Let i be a maximal bidder, and let $\delta_k(b) = H^{-i}_{m-k+1}(b) - H^{-i}_{m-k}(b)$. In any strictly-separating, continuous-bid equilibrium of the PAB auction, $\lim_{b \nearrow \bar{b}} \delta_k(b) > 0$ for all $k \in \{1, \ldots, m-1\}$.*

Proof. See Appendix C. □

Corollary 3 (Partial pooling in PAB)**.** *Contiunous-bid equilibria in the PAB auction exhibit partial pooling.*

[51] This is observed throughout the single-unit auction literature; with multiple units, the problem is exacerbated. For example, if each bidder has positive value for all m units ($m_i = m$ for all i), then when determining the bid for her $m - 2^{\text{nd}}$ unit a bidder must consider not only the possibility that any of her opponents receives 2 units and all the others receive 0 units (this is analogous to the single-unit case), but also to the possibility that any combination of two opponents each receives 1 unit while all others receive 0 units. This iso-allocation set makes standard no-gaps arguments inapplicable.

[52] This is in line with known results about multi-unit auctions, including Hortaçsu and Kastl [2012].

Remark 3. *With $n = 2$ bidders, all equilibria of the PAB auction exhibit partial pooling. When there are only two bidders in the model, the problem is analogous to a set of simultaneous two-bidder asymmetric auctions. Since the support of valuations is convex, standard results imply that bids are continuous in value, satisfying the antecedent of Lemma 3.*

Lemma 9 establishes the existence of (at least) two bidders who submit the same maximum bid for any unit. Compared to unit $\hat{k}$, there is a relatively small probability of having a high value for unit $\hat{k}+1$; however, *conditional* on having a high value for both units the bidder sees higher returns to increasing his bid for unit $\hat{k}+1$ than to increasing his bid for unit $\hat{k}$. This is in line with results in first-price auctions showing that distributional weakness leads to bid aggression. Since there is no value to bidding above the upper bound of the support of bids $\bar{b}$, strong incentives to bid on unit $\hat{k}+1$ induce a mass point at $\bar{b}$; this is our Lemma 10.

While on its face Lemma 10 appears to directly imply partial pooling — multiple types submit identical bids on single dimensions — this is not a complete characterization of partial pooling in the PAB auction. As is standard in other auction contexts, mass points in the bid distribution are not possible in equilibrium (above its lower bound). The construction in Lemma 10 contradicts this logic. Thus any separating equilibrium would have mass points at the upper bound of the bid distribution, which cannot occur in equilibrium. It follows that in any pure strategy equilibrium, partial pooling arises even though no bid occurs with strictly positive probability.

Partial pooling in the PAB auction arises due to the monotonicity constraint binding. Corollary 3 establishes this in some generality, but the intuition is most straightforward in the 2×2 balanced market case. Bidder i's bid for unit 1 is determined by her competition against bidder $-i$'s second-unit bid, and bidder i's bid for unit 2 is determined by her competition against bidder $-i$'s first-unit bid. In the order statistic model, the distribution of values for the first unit first order stochastically dominates the distribution of values for the second unit. If bids are determined independent of the monotonicity constraint, the "weakness leads to aggression" result from single-unit first price auctions implies that bidder i submits a more aggressive bid function for her second unit than for her first. When her values for these units are roughly equal, she will want to bid higher on her second unit than on her first, causing her monotonicity constraint to bind.

Remark 4. *The LAB auction admits a strictly-separating equilibrium, while equilibria in the FRB and PAB auctions exhibit partial pooling.*[53] *In the FRB auction, pooling arises due to low-value bidders submitting zero bids. In the PAB auction, pooling arises due to the monotonicity constraint binding when $v_{k+1} \approx v_k$.*

Proposition 8 (Inefficient equilibrium in PAB)**.** *All equilibria of the pay-as-bid auction are inefficient.*

Proof. In equilibrium, bid functions are either continuous or discontinuous. If they are continuous, equilibrium exhibits partial pooling (Lemma 3). Because the closure of the equilibrium market

[53] Continuous-bid equilibria in the PAB auction exhibit partial pooling. Well-behaved equilibriua in the FRB auction exhibit partial pooling.

clearing price set must be convex, when bids are discontinuous it must be that higher-value agents sometimes lose a unit to lower-value agents, implying inefficiency. In either case, outcomes are inefficient. □

6.3 Low-revenue equilibria and multiplicity

It has been noted (see, e.g., Engelbrecht-Wiggans and Kahn [1998], and Ausubel et al. [2014]) that the FRB auction frequently admits equilibria with arbitrarily small seller revenue. As implied by the proof of Lemma 8, all well-behaved equilibria in the FRB auction yield zero revenue with positive probability. We show now that the LAB auction does not admit any such equilibria.[54]

Proposition 9 (Zero-revenue equilibrium in FRB)**.** *The FRB auction always admits zero-revenue equilibria. The LAB auction never admits zero-revenue equilibria.*

Proof. Construction of low-revenue equilibria in a multi-unit setting is similar to that in a single-unit setting:[55] take numbers $(\tilde{m}_i) \in \mathbb{N}_0^n$ such that $\sum_{i=1}^{n} \tilde{m}_i = m$. Pick $\bar{s} \geq 1$ and let bidder i submit the bid

$$b^i(q; v) = \begin{cases} \bar{s} & \text{if } q \leq \tilde{m}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then the equilibrium price is always zero, independent of the bidders' private information; moreover, to win a greater quantity agent i must bid $\bar{s}$ for unit $\tilde{m}_i + 1$, obtaining weakly negative gross utility on this unit and incurring an additional payment of $\tilde{m}_i \bar{s}$. This is never utility-improving, hence these bid functions represent an equilibrium.

It is straightforward to show, by contradiction, that the LAB auction does not admit zero-revenue equilibria. Letting $q^i(v^i, v^{-i})$ be the equilibrium quantity allocation of agent i given value profiles v^i and v^{-i}, it is without loss of generality to assume that $q^i(v^i, v^{-i}) < m$ with positive probability.[56] Note that for almost all v^{-i} such that $q^i(v^i, v^{-i}) < m$, $b^i(q^i(v^i, v^{-i}) + 1; s_i) = 0$; furthermore, $b^{-i}(m - q^i(v^i, v^{-i}) + 1; v^{-i}) = 0$. Then by increasing her bid for units $q > q^i(v^i, v^{-i})$ to $\varepsilon > 0$, bidder i will incur an additional cost of at most $m\varepsilon$ but will win unit $q^i(v^i, v^{-i}) + 1$ with discretely positive probability. For ε sufficiently small this deviation is profitable, hence there is no low-revenue equilibrium. □

Proposition 9 establishes that the FRB pricing rule admits many zero-revenue equilibria, however the notion of "many" in this context is ill-defined. In particular, it is possible that the zero-revenue equilibria have measure zero in the set of all equilibria. This caveat aside, because the zero-revenue equilibria are simple focal points for collusive behavior, whether they are measurably present in the set of all equilibria does not affect the fact that they present a real practical concern.

[54] Our previous results are robust to the introduction of a strictly positive reserve price. In the (multi-unit) Vickrey auction a strictly positive reserve price — even infinitesimally small — is sufficient to eliminate zero-revenue equilibria; see, e.g., Blume and Heidhues [2004], and Blume et al. [2009].

[55] See, e.g., Milgrom [2004], pages 262–264.

[56] This follows from market clearing and the fact that we can focus on any particular agent.

7 Conclusion

We have defined a model of multi-unit auctions in which bidders have private values given by ordered draws from a single distribution. In this model, we show that the last accepted bid uniform-pricing rule induces bidding incentives analogous to those in a single-unit first price auction. We show that the last accepted bid auction can admit a tractable representation and can be both efficient and fully-revealing of bidders' private information. By noting the connection between bidding incentives in a single-unit first price auction and a multi-unit last accepted bid auction, we identify a new salient feature common to both auctions: in both auctions, bidders pay the highest market clearing price.

We compare the last accepted bid auction to the first rejected bid uniform price and the pay as bid auctions. We show that in each of these auctions bidder information is confounded in natural classes of equilibria. This further implies that these auctions are generally inefficient. We provide an additional construction which emphasizes that the first rejected bid auction always admits low-revenue equilibria, a phenomenon which cannot be sustained in the last accepted bid auction.

In symmetric single-unit auctions, efficiency and revenue typically move in opposite directions: when behavior is symmetric awarding the object to the highest bidder is efficient, so revenue can only be improved by conditionally not allocating the object (as with a reserve price). Thus revenue can increase as efficiency falls through nonallocation. In multi-unit auctions this connection breaks down. As we show in Section 6 inefficiency can arise by misallocation of goods. Then even in a symmetric equilibrium it is possible for revenue and efficiency to be simultaneously improved. We do not address this in this paper, but we believe the potential revenue gains of the last accepted bid auction over other multi-unit auction formats are worthy of future study.[57]

Taken as a whole, our results are strongly in favor of employing the last accepted bid pricing rule rather than the first rejected bid pricing rule when a uniform price auction is implemented. To date the literature has overlooked the possibility of a meaningful difference between the two; we show that this difference is real and has material implications in support of the last accepted bid auction.

References

Edward J Anderson, Pär Holmberg, and Andrew B Philpott. Mixed strategies in discriminatory divisible-good auctions. *The RAND Journal of Economics*, 44(1):1–32, 2013.

Olivier Armantier and Erwann Sbaï. Estimation and comparison of treasury auction formats when bidders are asymmetric. *Journal of Applied Econometrics*, 21(6):745–779, 2006. ISSN 1099-1255. doi: 10.1002/jae.875.

[57] In Section 6 we show that zero revenue occurs with positive probability in all equilibria of the first rejected bid auction, which does not establish a ranking in expected revenue. In Section 2 we show that all equilibria of the particular first rejected bid auction have zero expected revenue, which does not establish a ranking in general models.

Olivier Armantier and Erwann Sbaï. Comparison of alternative payment mechanisms for french treasury auctions. *Annals of Economics and Statistics/Annales d'Économie et de Statistique*, pages 135–160, 2009.

Leandro Arozamena and Estelle Cantillon. Investment incentives in procurement auctions. *Review of Economic Studies*, 71:118, 2004.

L. M. Ausubel, P. Cramton, Pycia M., Rostek M., and Weretka M. Demand reduction and inefficiency in multi-unit auctions. *Review of Economic Studies*, 81(4):1366–1400, 2014.

Kerry Back and Jaime F Zender. Auctions of divisible goods: On the rationale for the Treasury experiment. *Review of Financial Studies*, 6(4):733–764, 1993.

Patrick Bajari. Comparing competition and collusion: A numerical approach. *Economic Theory*, 18:187–205, 2001.

Andreas Blume and Paul Heidhues. All equilibria of the vickrey auction. *Journal of Economic Theory*, 114(1):170–177, 2004.

Andreas Blume, Paul Heidhues, Jonathan Lafky, Johannes Münster, and Meixia Zhang. All equilibria of the multi-unit vickrey auction. *Games and Economic Behavior*, 66(2):729–741, 2009.

Michal Bresky. Revenue and efficiency in multi-unit uniform-price auctions. *Games and Economic Behavior*, 82:205–217, 2013.

Sara Castellanos and Marco Oviedo. Optimal bidding in the mexican treasury securities primary auctions: results of a structural econometric approach. *Cuadernos de economía*, 45(131):3–28, 2008.

Sylvain Chassang and Juan Ortner. Collusion in auctions with constrained bids: Theory and evidence from public procurement. 2015.

Shuchi Chawla and Jason D Hartline. Auctions with unique equilibria. In *Proceedings of the fourteenth ACM conference on Electronic commerce*, pages 181–196. ACM, 2013.

Richard Engelbrecht-Wiggans and Charles M. Kahn. Multi-unit auctions with uniform prices. *Economic Theory*, 12:227–258, 1998.

Richard Engelbrecht-Wiggans and Charles M Kahn. Multiunit auctions in which almost every bid wins. *Southern Economic Journal*, pages 617–631, 2002.

Philippe Fevrier, Raphaële Préget, Michael Visser, et al. *Econometrics of share auctions*. 2002.

Milton Friedman. Single-price auction with multiple bids. *Wall Street Journal*, October 1991.

Daniel A Graham and Robert C Marshall. Collusive bidder behavior at single-object second-price and english auctions. *Journal of Political economy*, 95(6):1217–1239, 1987.

Samuel Häfner. Value bounds and best response violations in discriminatory share auctions. 2015.

Pär Holmberg. Supply function equilibria of pay-as-bid auctions. *Journal of Regulatory Economics*, 36(2):154–177, 2009.

Ali Hortaçsu and Jakub Kastl. Valuing dealers' informational advantage: A study of Canadian Treasury auctions. *Econometrica*, 80(6):2511–2542, 2012.

Ali Hortacsu and David McAdams. Mechanism choice and strategic bidding in divisible good auctions: an empirical analysis of the turkish treasury auction market. *Journal of Political Economy*, 118(5):833–865, 2010.

Ali Hortaçsu, Jakub Kastl, and Allen Zhang. Bid shading and bidder surplus in the us treasury auction system. *American Economic Review*, 108(1):147–69, 2018.

Boo-Sung Kang and Steven L Puller. The effect of auction format on efficiency and revenue in divisible good auctions: A test using Korean Treasury auctions. *The Journal of Industrial Economics*, 56(2):290–332, 2008.

René Kirkegaard. Asymmetric first price auctions. *Journal of Economic Theory*, 144:1617–1635, 2009.

Paul D. Klemperer and Margaret A. Meyer. Supply function equilibria in oligopoly under uncertainty. *Econometrica*, 57(6):1243–1277, 1989.

Albert S. Kyle. Informed speculation with imperfect competition. *The Review of Economic Studies*, 56(3):317–355, 1989.

Bernard Lebrun. First price auctions in the asymmetric n bidder case. *International Economic Review*, 40(1):125–142, February 1999.

Bernard Lebrun. Uniqueness of the equilibrium in first-price auctions. *Games and Economic Behavior*, 55:131–151, 2006.

Bernard Lebrun and Marie-Christine Tremblay. Multiunit pay-your-bid auction with one-dimensional multiunit demands. *International Economic Review*, 44(3):1135–1172, 2003.

Marco LiCalzi and Alessandro Pavan. Tilting the supply schedule to enhance competition in uniform-price auctions. *European Economic Review*, 49(1):227–250, 2005.

Alessandro Lizzeri and Nicola Persico. Uniqueness and existence of equilibrium in auctions with a reserve price. *Games and Economic Behavior*, 30(1):83–114, 2000.

Daniel Marszalec. The impact of auction choice on revenue in treasury bill auctions-an empirical evaluation. *International Journal of Industrial Organization*, 2017.

Eric Maskin and John Riley. Asymmetric auctions. *Review of Economic Studies*, 67:413–438, 2000.

Eric Maskin and John Riley. Uniqueness of equilibrium in sealed high-bid auctions. *Games and Economic Behavior*, 45:395–409, 2003.

David McAdams. Isotone equilibrium in games of incomplete information. *Econometrica*, 71(4): 1191–1214, 2003.

David McAdams. Monotone equilibrium in multi-unit auctions. *The Review of Economic Studies*, 73(4):1039–1056, 2006.

David McAdams. Adjustable supply in uniform price auctions: Non-commitment as a strategic tool. *Economics Letters*, 95(1):48–53, 2007a.

David McAdams. On the failure of monotonicity in uniform-price auctions. *Journal of Economic Theory*, 137(1):729–732, 2007b.

Paul Robert Milgrom. *Putting auction theory to work*. Cambridge University Press, 2004.

Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.

Michael Plum. Characterization and computation of nash-equilibria for auctions with incomplete information. *International Journal of Game Theory*, 20(4):393–418, 1992.

Marek Pycia and Kyle Woodward. Pay-as-bid: Selling divisible goods. 2017.

Philip J Reny. On the existence of monotone pure-strategy equilibria in bayesian games. *Econometrica*, 79(2):499–553, 2011.

William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16(1):8–37, March 1961.

Xavier Vives. Strategic supply function competition with private information. *Econometrica*, 79 (6):1919–1966, 2011.

James JD Wang and Jaime F Zender. Auctioning divisible goods. *Economic Theory*, 19(4):673–705, 2002.

Robert Wilson. Auctions of shares. *The Quarterly Journal of Economics*, pages 675–689, 1979.

Kyle Woodward. Strategic ironing in pay-as-bid auctions. 2016.

Kyle Woodward. Equilibrium and approximation in auction-like models. 2017.

A First-order conditions for leading example

A.1 Last accepted bid

Each bidder's utility can be expressed as

$$\begin{aligned} u^i\left(b^i;v^i\right) = {} & v_1^i F_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) + v_2^i F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) \\ & - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_1^i\right)\right)\right) b_1^i - 2b_2^i F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) \\ & - \int_{\varphi_1^{-i}\left(b_2^i\right)}^{\varphi_1^{-i}\left(b_1^i\right)} b_1^{-i}\left(v\right) dF_{(1)}\left(v\right). \end{aligned}$$

From here, it is straightforward to compute the model's first order conditions,

$$\begin{aligned} \frac{\partial}{\partial b_1^i} &: \left(v_1^i - b_1^i\right) dF_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) d\varphi_2^{-i}\left(b_1^i\right) - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_1^i\right)\right)\right); \\ \frac{\partial}{\partial b_2^i} &: \left(v_2^i - b_2^i\right) dF_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) d\varphi_1^{-i}\left(b_2^i\right) - 2F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right). \end{aligned}$$

Assuming a symmetric equilibrium many bid decorators can be dropped; substituting in for the known order statistic distributions ($F_{(1)}(x) = x^2$, $F_{(2)}(x) = 2x - x^2$) gives

$$\begin{aligned} 2\left(\varphi_1\left(b\right) - b\right)\left(1 - \varphi_2\left(b\right)\right) d\varphi_2\left(b\right) - \left(2\varphi_2\left(b\right) - \varphi_2\left(b\right)^2 - \varphi_1\left(b\right)^2\right) &= 0; \\ \left(\varphi_2\left(b\right) - b\right)\varphi_1\left(b\right) d\varphi_1\left(b\right) - \varphi_1\left(b\right)^2 &= 0. \end{aligned}$$

A.2 First rejected bid

Each bidder's utility can be expressed as

$$\begin{aligned} u^i\left(b^i;v^i\right) = {} & v_2^i F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) - 2\int_0^{\varphi_1^{-i}\left(b_2^i\right)} b_1^{-i}\left(v\right) dF_{(1)}\left(v\right) \\ & - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_2^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right)\right) b_2^i \\ & + v_1 F_{(2)}\left(\varphi_2^{-i}\left(b_1\right)\right) - \int_{\varphi_2^{-i}\left(b_2^i\right)}^{\varphi_2^{-i}\left(b_1^i\right)} b_2^{-i}\left(v\right) dF_{(2)}\left(v\right). \end{aligned}$$

From here, it is straightforward to compute the model's first-order conditions,

$$\begin{aligned} \frac{\partial}{\partial b_1^i} &: \left(v_1^i - b_1^i\right) dF_{(2)}\left(\varphi_2^{-i}\left(b_1^i\right)\right) d\varphi_2^{-i}\left(b_1^i\right); \\ \frac{\partial}{\partial b_2^i} &: \left(v_2^i - b_2^i\right) dF_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right) d\varphi_1^{-i}\left(b_2^i\right) - \left(F_{(2)}\left(\varphi_2^{-i}\left(b_2^i\right)\right) - F_{(1)}\left(\varphi_1^{-i}\left(b_2^i\right)\right)\right). \end{aligned}$$

B Proofs of equilibrium properties

Proof of Lemma 2. The first order condition for the constrained bid is

$$\sum_{y=k}^{k+a} \frac{\partial}{\partial b_y^i} u^i(b^i_{\{k,\ldots,k+a\}}; v^i) = 0, \tag{15}$$

or the sum of the unconstrained bid first order conditions. Note that the objective is quasi-concave in each b_y^i. At the largest unconstrained bid, b_l^i, the first-order conditions for the other bids cannot be positive, due to quasi-concavity, and given $b_l^i(v_l^i) > b_s^i(v_s^i)$ at least one is negative. Therefore, at b_l^i, the left-hand side of (15) is negative. A similar argument implies that the left-hand side of (15) is positive at b_s^i. □

Proof of Lemma 4. This is implied by continuity and differentiability of the bid distribution functions. These results are similar to the arguments familiar from the first price auction, but we reproduce them here due to the changes in the bidders' utility functions induced by shifting to a multi-unit model. We say that unit k is *opposed to* unit $m-k+1$, in the sense that agent i wins unit k if and only if agent $j \neq i$ wins unit $m-k+1$. Recall the separable utility representation for the LAB auction,

$$u^i(b;v) = \sum_{k=1}^{m} v_k H^{-i}_{m-k+1}(b_k) - \left(H^{-i}_{m-k+1}(b_k) - H^{-i}_{m-k}(b_k)\right) k b_k$$
$$- k \int_0^{b_k} x dH^{-i}_{m-k}(x) + (k-1) \int_0^{b_k} x dH^{-i}_{m-k+1}(x).$$ [58]

First, there are no gaps in equilibrium bid distribution functions. If there is a gap in H^{-i}_{m-k+1}, then a bid for agent i's unit k strictly inside this gap induces no additional winning probability but incurs additional expected costs (vis-à-vis bidding just above the lower bound). It follows that any gaps in H^{-i}_{m-k+1} are shared by the opposing distribution H^i_k. Since there is no probability gain within the gap, for a bid to be placed at the upper end of the gap there must be a mass point;[59] there are therefore identical mass points for the opposing units k and $m-k+1$. Identical mass points cannot arise for standard tiebreaking reasons, therefore this is not supportable in equilibrium.

Second, above the reserve price there are no mass points in equilibrium bid distributions (i.e., equilibrium bid distributions are continuous). Suppose that there is a mass point in H^{-i}_{m-k+1} at bid b, but no mass point in H^{-i}_{m-k}. Since bids are in general strictly below values[60] and there are no

[58] This expression appears to presuppose the differentiability of $H^{-i}_{k'}$ for all k', however it is a re-expression of one in terms of well-defined conditional expectations; since we establish that in equilibrium the bid distributions are continuously differentiable this expression is ultimately correct. We do not presuppose the correctness of this expression, and avoid this potential circularity in our formal arguments.

[59] This analysis ignores the possibility that the support of the bid distribution above the gap is left-open. For a bid sufficiently close to this upper endpoint, the arguments are the same.

[60] This somewhat obvious point is proved explicitly in an earlier version of this paper, and is familiar from results

gaps in the bid distributions, there is a value v such that $b_k^i(v) = b - \varepsilon$ for any $\varepsilon > 0$. For ε small enough, a slight increase to $\tilde{b}_k^i(v) = b + \varepsilon$ yields a discrete jump in expected utility; this implies that gaps exist in response to mass points, and we have already established that gaps cannot exist. Otherwise, suppose that there are mass points in both H_{m-k+1}^{-i} and H_{m-k}^{-i} at b, so that the above logic does not apply. However, if this is the case, then there is a mass point in H_{m-k}^{-i}, the unit opposed to bidder i's unit $k+1$. Then the previous argument holds unless there is also a mass point in H_{m-k-1}^{-i}, and so on. Since there are no mass points in the degenerate distribution H_0^{-i} — the $H_{m-\tilde{k}}^{-i}$ corresponding to $\tilde{k} = m$ — the original argument must hold for some unit, violating the no-gaps property established above.

Since equilibrium bid distributions are continuous and differentiable, the first order conditions must be satisfied in any equilibrium in separating strategies. □

Proof of Proposition 4. When bids are separating there can be no mass points in the bid distribution. This proof proceeds by ruling out gaps in first-unit bids, then successively ruling out differently-oriented kinks in bids.[61]

Recall from Appendix C that separable payoffs for bids for the two units are

$$u^1(b; v) = (v_1 - b_1) H_2(b_1) + \int_0^{b_1} H_1(x)\, dx,$$
$$u^2(b; v) = (v_2 - b_2) H_1(b_2) - \int_0^{b_2} H_1(x)\, dx.$$

The relevant probabilities are

$$H_1(x) = F_{(1)}(\varphi_1(x))^{n-1},$$
$$H_2(x) = (n-1) F_{(1)}(\varphi_1(x))^{n-2} F_{(2)}(\varphi_2(x)) \left(1 - F_{(1)}(\varphi_1(x))\right) + H_1(x).$$

Suppose that there is a gap (discontinuity) in a symmetric equilibrium first-unit bid function b_1. As is clear from the definition of unitwise utility, second-unit bids will never be placed in this gap; H_1 is constant on this gap so unitwise utility is strictly decreasing. Then if there is a gap in the first-unit bid there is a gap in the aggregate market clearing price distribution. As in other auction models, there is no incentive to bid just above the upper bound of the common gap, implying that this is not possible in an equilibrium without mass points.

Now suppose that there is a downward kink in b_1 at q, so that

$$\lim_{\varepsilon \searrow 0} \frac{b_1(q) - b_1(q-\varepsilon)}{\varepsilon} > \lim_{\varepsilon \searrow 0} \frac{b_1(q+\varepsilon) - b_1(q)}{\varepsilon}.$$

Since second-unit bids depend only on the first-unit bid function and there are no mass points, there must be a gap above q in the second-unit bid b_2. Then $F_{(2)} \circ \varphi_2$ is constant just above q, and

in single-unit auctions.

[61]In a monotone separating equilibrium, the quantity-monotonicity constraint will never strictly bind. This proof can be adapted to allow for binding monotonicity constraints, but it is not necessary to the point at hand.

first-unit bids depend only on $F_{(1)} \circ \varphi_1$. The downward kink in b_1 then implies either a mass point or a gap in b_1 just above q, which we have shown cannot arise.

Now suppose there is an upward kink in b_1 at q, so that

$$\lim_{\varepsilon \searrow 0} \frac{b_1(q) - b_1(q-\varepsilon)}{\varepsilon} < \lim_{\varepsilon \searrow 0} \frac{b_1(q+\varepsilon) - b_1(q)}{\varepsilon}.$$

Since the monotonicity constraint is not binding, this implies a gap in b_2 just above q. Per the previous argument regarding the downward kink, this is not possible in equilibrium.

Since there are no kinks in b_1 it is differentiable; this implies (via the same arguments as above) that b_2 is differentiable — nondifferentiabilities in b_2 will manifest in first-unit incentives, inducing gaps or mass points, which cannot arise. Then b_1 and b_2 are differentiable on their support, and the first order conditions must be satisfied in a symmetric separating equilibrium. □

C Proofs of information pooling properties

Proof of Lemma 6. We analyze each auction in turn. Note that it is without loss in each case to consider the agent as bidding for all m available units, with the constraint that she has zero value for units $k > m_i$.

FRB. Utility is written as

$$\begin{aligned} u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} &\left(\sum_{k'=1}^{k} v_{k'} - k b^i_{k+1}\right) \Pr\left(b^{-i}_{m-k} \geq b^i_{k+1} \geq b^{-i}_{m-k+1}\right) \\ &+ \left(\sum_{k'=1}^{k} v_{k'} - k \mathbb{E}\left[b^{-i}_{m-k+1} \middle| b^i_k \geq b^{-i}_{m-k+1} \geq b^i_{k+1}\right]\right) \\ &\times \Pr\left(b^i_k \geq b^{-i}_{m-k+1} \geq b^i_{k+1}\right). \end{aligned}$$

The relevant probabilities are

$$\begin{aligned} \Pr\left(b^{-i}_{m-k} \geq b^i_{k+1} \geq b^{-i}_{m-k+1}\right) &= H^{-i}_{m-k+1}\left(b^i_{k+1}\right) - H^{-i}_{m-k}\left(b^i_{k+1}\right), \\ \Pr\left(b^i_k \geq b^{-i}_{m-k+1} \geq b^i_{k+1}\right) &= H^{-i}_{m-k+1}\left(b^i_k\right) - H^{-i}_{m-k+1}\left(b^i_{k+1}\right). \end{aligned}$$

Algebraic manipulation gives a separable utility form of

$$\begin{aligned} u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} & v_k H^{-i}_{m-k+1}\left(b^i_k\right) - \left(H^{-i}_{m-k+2}\left(b^i_k\right) - H^{-i}_{m-k+1}\left(b^i_k\right)\right)(k-1)\, b^i_k \\ &- k \int_0^{b^i_k} b\, dH^{-i}_{m-k+1} + (k-1) \int_0^{b^i_k} b\, dH^{-i}_{m-k+2}. \end{aligned}$$ [62]

LAB. Utility is written as

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m}\left(\sum_{k'=1}^{k} v_{k'} - kb_k^i\right)\Pr\left(b_{m-k}^{-i} \geq b_k^i \geq b_{m-k+1}^{-i}\right)$$
$$+\left(\sum_{k'=1}^{k} v_{k'} - k\mathbb{E}\left[b_{m-k}^{-i}\middle| b_k^i \geq b_{m-k}^{-i} \geq b_{k+1}^i\right]\right)$$
$$\times \Pr\left(b_k^i \geq b_{m-k}^{-i} \geq b_{k+1}^i\right).$$

The relevant probabilities are

$$\Pr\left(b_{m-k}^{-i} \geq b_k^i \geq b_{m-k+1}^{-i}\right) = H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_k^i\right),$$
$$\Pr\left(b_k^i \geq b_{m-k}^{-i} \geq b_{k+1}^i\right) = H_{m-k}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_{k+1}^i\right).$$

Algebraic manipulation gives a separable utility form of

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m} v_k H_{m-k+1}^{-i}\left(b_k^i\right) - \left(H_{m-k+1}^{-i}\left(b_k^i\right) - H_{m-k}^{-i}\left(b_k^i\right)\right) kb_k^i$$
$$- k\int_0^{b_k^i} bdH_{m-k}^{-i} + (k-1)\int_0^{b_k^i} bdH_{m-k+1}^{-i}.$$

PAB. This is essentially trivial. Note that utility has a naturally separable form,

$$u^i\left(b^i, b^{-i}; v\right) = \sum_{k=1}^{m_i}\left(v_k - b_k^i\right) H_{m-k+1}^{-i}\left(b_k^i\right).$$

□

Proof of Lemma 7. Note that strict separation implies that strategies are strictly monotone in value. If strategies cannot be separated as in the statement of the Lemma, the monotonicity constraint must be binding.[63]

Suppose first that bids are continuous in value. If the bid profile cannot be written as a product of independent dimensional bids, the monotonicity constraint must be binding over some product of nondegenerate intervals $[\underline{b}_k, \overline{b}_k) \times \cdots \times [\underline{b}_{k'}, \overline{b}_{k'})$. Because bids are continuous in value, bids are not invertible on this range; since this range has positive measure (and can be expanded to account

[62]This form is a convenient symmetric shorthand, but H_{m+1}^{-i} is ill-defined. Since this term is in $[0, 1]$ and is always premultiplied by $(1-1) = 0$, the exact specification is irrelevant.

[63]Due to standard peculiarities of measure zero, this statement is true when constrained to left-continuous bid functions but may fail in the presence of arbitrary discontinuities. It is straightforward to show that any monotone strictly separating equilibrium is incentive-equivalent to an equilibrium which is left-continuous in value, thus this statement is more or less without loss of generality.

for higher and lower units for which the monotonicity constraints are not binding) it follows that equilibrium is not strictly separating, a contradiction.

Now suppose that the monotonicity constraint is binding at a point at which bids are discontinuous in value. Because dimensional utilities satisfy increasing differences and are continuous in value, there is a neighborhood below this point on which the monotonicity constraint is binding and bids are locally continuous. Then the above argument holds. □

Proof of Lemma 8. The unitwise utility function in the FRB auction can be expressed as

$$u_k^i\left(b_k^i, b^{-i}; v_k\right) = \left(v_k - b_k^i\right) H_{m-k+1}^{-i}\left(b_k^i\right) - (k-1)\int_0^{b_k^i} H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\, dx.$$

Since for any unit the agent has the option of bidding 0 and obtaining (at worst) zero utility, $u_k^i(b_k^i(v), b^{-i}; v_k) \geq 0$ whenever b_k^i is a best response bidding function. As established in Lemma 7, if equilibrium does not exhibit partial pooling, $b_k^i(v) \equiv b_k^i(v_k)$. It follows that in an equilibrium without partial pooling,

$$\frac{v_k - b_k^i(v_k)}{k-1} \geq \frac{\int_0^{b_k^i(v_k)} H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\, dx}{H_{m-k+1}^{-i}\left(b_k^i(v_k)\right)}.$$

Strict separation, well-behavedness, and best-responseness require that $b_k^i(v_k) > 0$ whenever $v_k > 0$, that bids are dense near 0, and that $b_k^i(0) = 0$.[64] Then in the limit, for all $k > 1$,

$$\lim_{b \searrow 0} \frac{\int_0^b H_{m-k+2}^{-i}(x) - H_{m-k+1}^{-i}(x)\, dx}{H_{m-k+1}^{-i}(b)} = 0.$$[65]

When equilibrium is well-behaved and arbitrarily differentiable, for a set of relevant $t \in \{0, 1, \ldots, \bar{t}\}$ l'Hôpital's rule implies

$$\lim_{b \searrow 0} \frac{d^{(t)} H_{m-k+2}^{-i}(b) - d^{(t)} H_{m-k+1}^{-i}(b)}{d^{(t+1)} H_{m-k+1}^{-i}(b)} = 0.$$[66]

Bid monotonicity requires that $b_{k'}^j(v) > b_{k'+1}^j(v)$, and hence by the nature of the order statistic

[64] In a working version of this paper we provide arguments that these statements continue to hold in the presence of a reserve price $r > 0$.

[65] Technically only a weak inequality, ≤ 0, is required. Given the relationship between H_{m-k+2}^{-i} and H_{m-k+1}^{-i} it is straightforward to show that strict inequality cannot be satisfied.

[66] This limit makes clear the hidden role of the assumption that $m \geq 2$ units are available. When $m = 1$, $H_{m-k+2}^{-i} = 0$, invalidating this proof approach. This is to be expected, since with $m = 1$ unit available the FRB auction is equivalent to a second price auction, which admits a well-behaved, separating, truthful equilibrium.

model there is some t such that

$$\lim_{b\searrow 0}\left|d^{(t)}H_{m-k+2}^{-i}(b)\right|>\lim_{b\searrow 0}\left|d^{(t)}H_{m-k+1}^{-i}(b)\right|=0.$$

At this t, well-behavedness requires that $\lim_{b\searrow 0}|d^{(t+1)}H_{m-k+1}^{-i}(b)|\geq 0$ is finite, hence the limit is strictly positive, contradicting strict separation. □

Proof of Lemma 9. Let $\bar{b}_k^i$ be bidder i's maximum bid for unit k; without loss of generality, this is $\bar{b}_k^i=b_k^i(\bar{v})$. Suppose that $\bar{b}_k^i\neq\bar{b}_{m-k+1}^{-i}$, and without loss of generality assume that $\bar{b}_k^i>\bar{b}_{m-k+1}^{-i}$. Then anytime bidder i submits a bid $b\in(\bar{b}_{m-k+1}^{-i},\bar{b}_k^i]$, she wins unit k with probability 1; she could reduce her bid without affecting her winning probability, improving her utility. Then $\bar{b}_k^i=\bar{b}_{m-k+1}^{-i}$ for all units k.

Bid monotonicity requires that $\bar{b}_k^i\geq\bar{b}_{k'}^i$ for all $k'\geq k$. Then

$$\bar{b}_k^i\geq\bar{b}_{k'}^i=\bar{b}_{m-k'+1}^{-i}\geq\bar{b}_{m-k+1}^{-i}=\bar{b}_k^i.$$

Then $\bar{b}_k^i=\bar{b}_{k'}^i$ for all $k'\geq k$, and bidder i's maximum bid is independent of the unit she is bidding for. Since this maximum bid is equal to bidders' $-i$ maximum bid for the complementary unit, the maximum bid is independent of unit and agent. □

Proof of Lemma 10. If bidder i's best-response bid function for unit k, $b_k^\star$, is continuous, it must be that H_{m-k+1}^{-i} is continuous; moreover, where H_{m-k+1}^{-i} is not differentiable it has a "downward" kink. Following the first order conditions of the model, for any $v\in(0,1)$ it must be that either

$$b_k^\star(v)>b_{k+1}^\star(v)\,,\ \text{or}\ d_+H_{m-k}^{-i}\left(b_k^\star(v)\right)<d_+H_{m-k+1}^{-i}\left(b_k^\star(v)\right)\ \ \text{(or both).}^{67}$$

Since $b_{\tilde{k}}^\star$ is continuous for each $\tilde{k}$ (by assumption), whenever the first inequality holds it must hold over an interval. Note that is must be that there is some v for which the first inequality holds; otherwise $b_k^\star(v)=b_{k+1}^\star(v)$ for all v, implying the second inequality for all $b\in[0,\bar{b}]$. Then there is a mass point in H_{m-k}^{-i} at $\bar{b}$, which is inconsistent with i submitting a continuous bid function.[68]

Let v be such that $b_k^\star(v)>b_{k+1}^\star(v)$. Appealing to incentive compatibility and first-order dominance,

$$\begin{aligned}\left(v-b_k^\star(v)\right)H_{m-k+1}^{-i}\left(b_k^\star(v)\right)&\geq\big(v-b_{k+1}^\star(v)\big)H_{m-k+1}^{-i}\big(b_{k+1}^\star(v)\big)\\&>\big(v-b_{k+1}^\star(v)\big)H_{m-k}^{-i}\big(b_{k+1}^\star(v)\big)\\&\geq\left(v-b_k^\star(v)\right)H_{m-k}^{-i}\left(b_k^\star(v)\right).\end{aligned}$$

[67] The use of $b_k^\star$ (instead of $b_{k+1}^\star$) is irrelevant here, and is used solely to ensure that attention is focused on a single bid. It is also sufficient to consider only right derivatives, which are finite at all relevant points (otherwise a slight increase in bid would trivially be profitable, and would be feasible since bids are necessarily below values whenever right derivatives are nonzero).

[68] Either i's bid function is discontinuous, or the high bid is such that $\bar{b}=1$. In this latter case, the existence of a mass point implies some of i's opponents are bidding above their values and winning with positive probability, which is not a best response in the PAB auction.

These inequalities imply

$$\frac{H_{m-k+1}^{-i}\left(b_k^\star(v)\right)}{H_{m-k}^{-i}\left(b_k^\star(v)\right)} > \frac{H_{m-k+1}^{-i}\left(b_{k+1}^\star(v)\right)}{H_{m-k}^{-i}\left(b_{k+1}^\star(v)\right)}.$$

Continuity and maximality imply that there is $\tilde{v}$ with $b_{k+1}^\star(\tilde{v}) = b_k^\star(v)$. Then

$$\frac{H_{m-k+1}^{-i}\left(b_k^\star(\tilde{v})\right)}{H_{m-k}^{-i}\left(b_k^\star(\tilde{v})\right)} \geq \frac{H_{m-k+1}^{-i}\left(b_{k+1}^\star(\tilde{v})\right)}{H_{m-k}^{-i}\left(b_{k+1}^\star(\tilde{v})\right)}.$$

Let $I(b)$ be the interval over which $b_k^\star(v') > b_{k+1}^\star(v')$,

$$I(b) = \begin{bmatrix} \inf\left\{b_k^\star(v') : b_k^\star(\tilde{v}') > b_{k+1}^\star(\tilde{v}')\,\forall\tilde{v}' \in (v', v]\right\}, \\ \sup\left\{b_k^\star(v') : b_k^\star(\tilde{v}') > b_{k+1}^\star(\tilde{v}')\,\forall\tilde{v}' \in [v, v')\right\} \end{bmatrix}.$$

The preceding inequalities and standard sequential arguments imply that the difference between the CDFs H_{m-k+1}^{-i} and H_{m-k}^{-i} is maximized at the interval's right endpoint,

$$\begin{aligned} &H_{m-k+1}^{-i}\left(\min I\left(b_k^\star(v)\right)\right) - H_{m-k}^{-i}\left(\min I\left(b_k^\star(v)\right)\right) \\ &\quad < H_{m-k+1}^{-i}\left(\max I\left(b_k^\star(v)\right)\right) - H_{m-k}^{-i}\left(\max I\left(b_k^\star(v)\right)\right). \end{aligned}$$

The right endpoint of the interval is either the left endpoint of another interval on which $b_k^\star(\tilde{v}) > b_{k+1}^\star(\tilde{v})$, or of an interval on which $d_+H_{m-k}^{-i}(b) < d_+H_{m-k+1}^{-i}(b)$. In the latter case, the difference between the two CDFs is again maximized at the right endpoint of the subsequent interval (and the former case is as analyzed above). In either case, the difference between the CDFs at the right endpoint is increasing in the location of the interval. Since $H_{m-k+1}^{-i} \succeq_{\text{FOSD}} H_{m-k}^{-i}$, it follows that $H_{m-k}^{-i}(\bar{b}) < H_{m-k+1}^{-i}(\bar{b})$ and hence H_{m-k}^{-i} has a mass point at $\bar{b}$. □

Proof of Corollary 3. This is a direct consequence of Lemmas 9 and 10. At $\bar{b}$, $H_{m-k+1}^{-i}(\bar{b}) > H_{m-k}^{-i}(\bar{b})$, contradicting best response behavior. □